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Optimization of facility layout

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OPTIMIZATION OF FACILITY LAYOUT

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TABLE OF CONTENTS

Page

INTRODUCTION

"Layout Planning

-Placing the right equipment
-Coupled with the right method
-In the right place to permit the processing of
products or units in the most effective manner
-Through the shortest possible distance
-In the shortest possible time." Author Unknown.

The problem of proper layout design for most efficient operation is one which all organizations face. Within industry, government, or other classification, whenever individuals must interact for the purpose of product or service production they are faced with the ever increasing challenge of productivity and competition. They are forced to consider the proper arrangement of their work centers as one of the ingredients in meeting this challenge.

As technology advances, so the ability to meet the challenges is broadened in scope. And so it should be with the problem of layout design. The computer has opened many doors because of its ability to handle tedious computations and evaluations very rapidly, far more rapidly than is practical by human brain power with the aid of pencil and paper. It is this characteristic that makes the computer important to the more successful attempts to improve layout design techniques.

However, computer technology coupled with the mathematicians brain power is not yet to the point where it is practical to deterministically evolve an optimum layout. Heuristic procedures have been documented which provide layouts that, by one measure of efficiency or another, tend toward the optimum. But only by chance, and a very slight one in most cases, do they present an optimum. What is more disconcerting, these procedures do not include an effective means of determining how close to an optimum the best of the heuristically developed layouts might be.

This problem can be overcome if the various products of existing heuristic layout procedures can be shown to exhibit a regular behavior such that pertinent statistics can be developed. Such an analysis would be valuable in estimating the efficiency of the heuristic output and supply the needed ingredient to make quantitative facility layout a useful tool and broaden the scope of the layout designer.

Presented in subsequent sections is an analysis which is based on a particular procedure initially developed by F. S. Hillier (24). The existence of a regular behavior pattern of the procedure output is demonstrated herein with resulting estimates of the optimum layout cost in terms of minimum traffic intensity, the probability of finding a better layout than the best generated by the procedure, and a measure of how much additional resource should be expanded to search for the optimum. These estimates result from an application of the theory of extreme values which heretofore has been found to be of greatest value in reliability estimation. It is believed that this present application will indicate many other applications in the area of industrial management.

BACKGROUND AND CHARACTERISTICS OF FACILITY LAYOUT PROBLEMS

> "When we know exactly what is and also exactly what ought to be, we are able to establish a direct efficiency relation. By appropriate comparisons of what is and what ought to be, efficiencies, both ideal and practical, can be established. To ascertain what is, to establish standards and to bring the actual up to the standard, requires all sorts of knowledge, experience, efforts, methods, devices, accumulated during the past ages or newly solved." (Emerson (12, p. 23))

An industrial operation may be regarded as a total system into which raw material is introduced and on which the factors of production, i.e., men, machines, management, methods, etc., operate to turn out a finished product to be placed on the market. How the system operates is, among other factors, a function of the number of different product types produced and the quantities required of each. The relationship between each of the elements of the system then is one of the key factors to be considered in determining "what ought to be". This relationship, more frequently referred to as the layout or the arrangement of the factors of production, is a problem that has had to be dealt with from the beginning of mans effort to improve his lot.

The layout may be a very bad one and the system may continue to operate, but not well. The concept of efficiency is introduced by the fact that to continue to operate at all, "what is" must come closer to "what ought to be" or else the raw materials would generate a greater good by going to some other use. Thus the alternative cost aspects, whether it be in dollars or some other real or abstract notation, becomes the criteria of concern.

Efficiency, by itself, is as meaningless as it is dimensionless. It is only when it is considered against an objective that it takes meaning as a measure. In the industrial sense, the layout is measured most often by a cost criterion with a minimization of cost as its objective. The concern for cost expressed in terms of travel distance or travel time for material handling is in reality an attempt to control costs by minimizing the overhead costs associated with travel balanced against the necessity of having some travel in order to carry out the production function at all, the real objective.

The costs may be associated with the use of space which in turn would minimize overhead costs associated with non-productive plant, again in an environment that some such plant is necessary for the fulfillment of the production objective. Or again the cost concept is employed in the minimization of product or material delays by a well designed layout so as to minimize the costs of capital tied up in the delay. In this sense the attention to layout planning of production facilities is done as a means of reducing system costs. In many industrial concerns, this attention to layout planning is crucial to its continued operation.

Traditionally, the problem of layout planning has not been approached from the view of a single large system which is to be integrated. Rather the system has been segmented by functional department, component manufacture, or other criteria. The cost control considerations have been related to these sub-systems. The underlying assumption has been that the sum of the resulting cost minima or the costs which are believed to be minimum as achieved by some specified mode of operation, result in a minimization of the total. This is often not the case. A major reason for the traditional approach has been the lack of an effective or economical means whereby a total system can be designed without segmenting. A second difficulty has been in the identification of a measure of effectiveness which, when optimized, is equally applicable to all segments of the system.

The layout planning problem can perhaps best be identified by a multiple classification schematically presented in Figure 1. This classification structure assumes that the key functional factor is the final output or product of the facility for which the layout planning is being done. Use of the final product as the key link is not an unrealistic assumption as the creation of the product is the sole reason for the facility to exist at all. As such it does not seem inappropriate to assume the product to be the keystone on which the layout planning is developed.

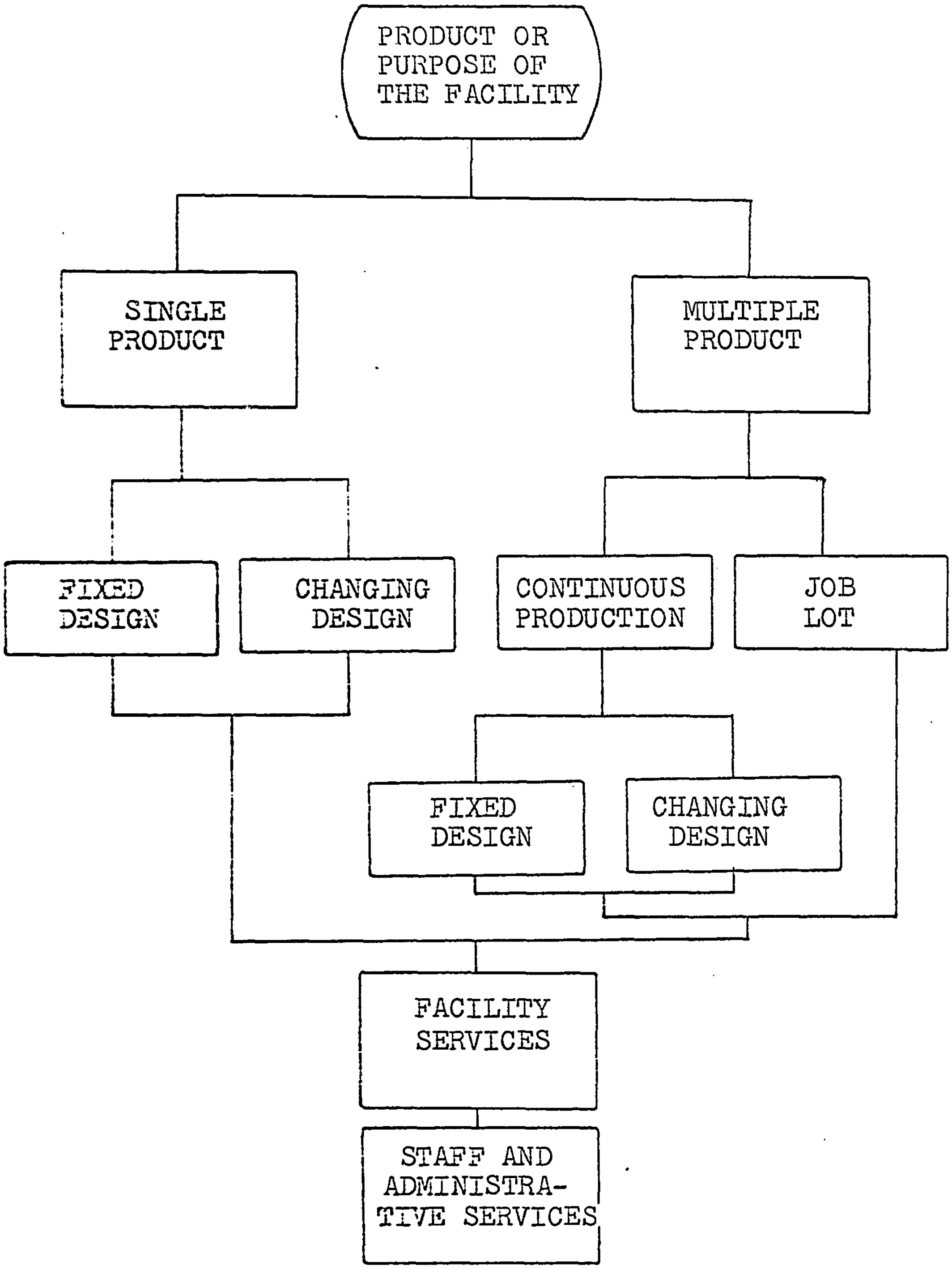

Figure 1. Classification of facility layout problems

Single Product

The least complex industrial process from a layout planning viewpoint would involve the production of a single product of fixed design or in high volume. Once the product design and process specifications have been established, the flow-through concept of facility layout can be applied to the resulting work centers. It is an over simplification to refer solely to the flow-through concept at this point as the interactions among the product design, process design, work center design, process specifications, and the cost associated with each are inescapable. The "optimum" combination of these interactions is usually determined by a trial and error process implied by the terms "pilot plant" or "development".

The fixed design and high production volume, inasmuch as both imply relatively long periods of continuous operation, are considered to affect the results in the same way. As such the requirement for replanning of the layout would only occur when the cost relationships among the factors of production are significantly altered. This alteration may result from plant obsolescence, availability of new processing techniques, or general economic shift which may make a new layout desirable from a cost minimization standpoint. The combination of fixed design and low volume is not, in the macro sense, significantly different from the single product changing design condition.

The single product with changing design refers to the type of product for which frequent changes of design are encountered. This effect is similar to that which would be the result of multiple product produced sequentially, i.e., the tooling up and running of a complete requirement of one product and then retooling for the second product, etc., and where the production requirements are low in volume. The result of these conditions is short run, frequent change situations. Design changes necessitated by customer preference changes, development of new materials or processes resulting in product improvements, and technological advances tend to be the causes of the type of design change referred to here. Under these situations new layouts may be designed frequently under the cost minimization concept. In many of these situations it is conceivable that the cost minimization principle might better be served by not responding to the frequent product changes. The criterion guiding the layout may better be considered over a longer time span encompassing several product changes. To do this of course increases the scope of the problem and adds additional variables, the interaction of which may complicate the planning beyond the capabilities of existing methods and techniques.

Many examples of frequent changes of this type are available in the electronic industry. Here assembly benches are portable and power conduit runways are so constructed that work centers can be quickly rearranged to meet the rapidly changing demands for new models of specialized products requiring, in turn, component design changes.

Multiple Product

A separate classification of the layout problem is that involving multiple products. Included in this classification would be the single product with many components and subassemblied as well as the job lot type of production operation.

The sub-class of multiple product which is identified by characteristics of prescheduled production, i.e., where production is to stock, provides some unique problems in addition to those similar in nature to the single product, changing design sub-class. While layout changes are frequently considered necessary within a work center or assembly line, the cost of the rearrangement may be included in the set-up cost consideration of an economic lot quantity determination. This would be particularly true in the multiple product, fixed design case where the change-over costs are somewhat more predictable. Again from a total optimization view, a layout considering all products may be preferable, from a cost minimization objective, to a relayout every time a scheduled change occurs. The nature of the costs inherent in the objective is crucial here. There very likely will be a hierarchy of goals, many of which are likely to be conflicting if the total objective is not carefully defined. For example, the layout providing a minimum of production cost may be far from optimum in meeting the goals identified as the minimization of delivery times or the maximization of plant utilization.

The changing design sub-classification adds greater uncertainty into the forecast of future costs. This sub-class would effectively be identical to the shortening of the time horizon over which a layout may be effective. The relayout effort will frequently be necessary in view of the complexity in developing the long-term alternatives that might satisfy the objectives. Consequently the set of alternatives from which to choose that one which will optimize the layout may only consider the short run optimization.

The job lot type of operation differs from the pre-scheduled production operation in that the customers product design and order quantity rather than the economic lot quantity determination is controlling. Thus the factors of length of run and frequency of design change become, to a much greater degree, externally dictated and controlled. In this situation there has traditionally been a greater effort to "optimize" on a total system base by functional groups of machine tools or work centers. However, the location of these groupings relative to the others have resulted largely from trial and error analyses. If the functional grouping is treated as a single work center, the problem has many characteristics in common with the multiple product, advance schedule, changing design case in which the "optimum" location of each center relative to every other is sought.

Some quantified techniques aimed at the job shop problem have appeared in the literature within the past few years. These techniques, essentially simulation approaches, have been quite cumbersome to handle. As the number of work centers increase, many of the techniques have become impractical if not impossible to handle.

Joint Use Facilities

The joint use of facilities where multiple products are produced permits a greater opportunity for overall optimization which may be compared to the sum of the optima of the production arrangements for each product. In a sense the job shop may represent a complete joint use of each functional group by all products but in different sequential order. As such, the greater the opportunity for the facilities to be used jointly, the better the problem may respond to a long term, single product analysis.

Plant Services, Staff and Administrative Services

The problems of incorporating the plant services and the staff and administrative service facilities into the "optimum" is difficult. The requirements for plant service follow diverse paths for the single and multiple product cases. Within each case the difficulties stem from a lack of common criteria with the production facilities on which an optimization may be based. Some of the plant service requirements may be treated as additional work centers, e.g., rest room facilities which are determined by the number of employees, or scheduling functions which may be located to minimize total distance from other work centers. In treating the service facilities in this way, however, care should be taken in the weighting given the distance factor since the distance to the service facility may have greater or lesser importance than the distance between production centers.

Other service facilities such as some maintenance, general plant janitorial, general warehousing, etc., may be located more remotely because of the independent nature of the activity or the satisfactory linking by remote communication media. While these could be included in the minimization of load-density between the pertinent centers, the load may be product in one case and personnel in another. Thus the load-density factor must be weighted in order for the minimization to be meaningful.

The lack of commonality of criteria between production centers and staff and administrative facilities is an even greater problem. However, since these criteria are so diverse and the objectives of each are separate and distinct sub-objectives of the total plant or industry objectives, the treatment of staff and administrative facilities as a separate layout problem does not seem inappropriate.

The above discussion suggests that a generalized model for the design of a facility layout would serve to compare total system optimization to the sum of sub-system optima. In applications, a major difficulty initially apparent is the absence of common criteria by which the comparison may be made.

PRESENT TECHNIQUES OF FACILITY LAYOUT

Qualitative Models

The use of models of one form or another has long been the means of accomplishing a layout design. Initial steps for the construction of the model are to make a qualitative analysis of the product flow. Where multiple products exist, either the flow patterns are weighted by the relative production volumes or rearrangements are considered when the product shift is made. While by definition this approach is an attempt at optimization, it is necessarily limited by the number of interrelationships humanly possible to comprehend. Hence, the segmenting to sub-system designs of a size that could be qualitatively analyzed is a practical means to solution.

Some techniques, such as represented by process flow charts, provide an analogue type of model by which flow patterns may be examined. The analogue type of analysis frequently coexists with another type of analogue or iconic model to augment and clarify the analysis. These secondary models vary from a set of templates arranged and rearranged on a two dimensional drawing to a full three dimensional representation of the work centers to be arranged and are used to give a better means of evaluating the qualitative analysis. Models are used because of the ease in manipulation for effecting changes prior to the production line changes and to permit better coordination and better planning.

Since the creation of a model which can be manipulated for the purposes of better planning, prediction and control is the prime objective of these efforts, a more general model which can more easily be manipulated and which can represent a larger number of interrelationships would be preferable. These characteristics are usually found in the mathematical model although initial design and construction is frequently more difficult.

Quantitative Models

While the literature has revealed a number of attempts to quantify the area of plant layout, they have basically fallen into the heuristic type of analysis. Simon and Newell (43) have described the heuristic process as being a particular approach to problem solving and decision making by use of logic and common sense derived by introspection. In essence the heuristic approach applies selective routines to reduce the size of a problem. Thus a complex production problem may be treated by reducing the total system to a series of simpler problems. As such, it is an approach used to simulate situations which do not lend themselves to mathematical analysis. Heuristic approaches do not necessarily produce an optimal result but rather serve to "investigate the relative goodness of various strategies subject to specific constraints" (Starr (45, p. 183)). A further point regarding the optimality of the results is made by Starr (45, p. 291):

> "Can an optimal layout really be found? At the present time, it is nonoperational to talk about an optimal arrangement. There are so many possible variations, and usually no way to search through them all. As the production process approaches total mechanization and, ultimately, complete automation, then, technological constraints begin to operate and the notion of an optimal layout becomes more tenable. For the general case it is desirable to talk about a satisfactory layout, or perhaps, just a good one."

As a result, the existing quantitative efforts have been primarily an extension of the qualitative techniques and most of the problems associated with the analysis of a complex system remain. Several of the quantitative efforts at plant layout are indicated below as examples of the quantitative efforts currently available.

Noy's sequence demand

Peter C. Noy (36) discusses a technique that considers the sequence of operations on essentially a multiple product layout. Within a limited sequence of work centers, several products may be processed through all of them but not necessarily in the same order. The technique considers that all work centers are the same size and that any area adjustment may be made after the arrangement has been determined. Then, considering some common measure of transfer for all products through the entire sequence, the weighted mean position for each operation is determined. For example, if product A has

10 equivalent units of output and work center 1 is the first of the product manufacturing sequence while product B with 40 equivalent units requires work center 1 in the third position of its sequence then the weighted average[1] of 2.4 indicates that work center 1 should be in position 2 or 3 in the system sequence.

The assumption of a common denominator limits the technique to a sub-system analysis. Secondly the assumption of expanding or condensing space to compensate for unequal size of work center may completely destroy the location advantage determined on the basis of the density criterion. The complexity of the sub-system would not have to be very great before several work centers are vieing for the same sequence position. In this case no decision base is provided at all.

Wimmert's technique for nondirectional sequence demand

A technique presented by R. J. Wimmert (49) considers for its criteria a volume of demand between pairs of work centers. By developing a matrix of location combinations on machine combinations and filling the matrix cells with demand volume values a set of logically determined decision rules can be determined. An example Wimmert uses to illustrate his

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[1] $\frac{(10)(1) + (40)(3)}{50} = 2.6.$

concept involves arranging a turret lathe, a milling machine, an inspection station, and a drill press among four possible locations 1 - 4. The matrix of machine combinations and location combinations is illustrated in Table 1.

Table 1. Machine and location combinations for the Wimmert technique

Machine Combin-ation	Trips per mo.	Locations 1-2 44	1-3 54	2-3 62	3-4 88 ft.	2-4 140	1-4 142
TL-DP	380	16720	20570	23560	33440	53200	53960
DP-MM	305	13420					
TL-MM	240	10560					
TL-Insp.	165	7260					
MM-Insp.	95	4180		X_{ij} = (ft.)(trips/mo.)			
DP-Insp.	75	3300					

With the matrix arrayed with the distance monotonically nondecreasing to the right and the trips per month monotonically nonincreasing downward, the greatest value will be in the upper right and the smallest in the lower left. The logic pattern, which Wimmert defines more rigorously with symbolic notation, follows that the highest value cannot be a possibility for a minimized set of volumes. Therefore, the combinations of TL and DP on 1 and 4, which also implies MM and Insp. on 2 and 3, are eliminated. The next highest value occurs in the TL-DP on 2-4 intersection which then eliminates TL-DP on 2-4 and MM-Insp. on 1-3. The next highest value occurs on the intersection of DP-MM on 1-4 which eliminates DP-MM on 1-4 and TL-Insp. on 2-3. By this sequential elimination, all values are eliminated until only one intersection in each row and column remains with no duplications between rows and columns. This example results in the intersection set shown in Table 2.

Table 2. Resulting layout from the Wimmert example

TL-DP	on 1-3
DP-MM	on 1-2
TL-MM	on 2-3
TL-Insp	on 3-4
MM-Insp	on 2-4
DP-Insp	on 1-4

This set therefore implies DP in location 1, TL in 3, MM in 2 and Insp. in 4. The obvious disadvantage to this technique is again the cumbersome size of the analysis as the system analyzed becomes larger. The number of rows and columns in the matrix given n work centers is $\frac{n!}{2!(n-2)!}$. If n is 10, not at all unreasonable, the matrix is 45 x 45. If n grows to 50 which is still within the scope of a feasible problem, the matrix becomes 1225 x 1225.

Conway and Maxwell (8) take issue with the Wimmert technique by asserting that the principal theorem on which the technique is based is incorrect. Thus the technique gives results which may not be optimum with respect to demand volume.

Computerized relative allocation of facilities technique (CRAFT)

Buffa, Armour, and Vollmann (6) have developed a computerized method of attempting to design a facility layout which will minimize the material-handling costs. These costs, like the Wimmert criteria, are a function of the distance between work centers, the frequency of movement of material between the centers, and the cost involved in each move. The process is essentially one of exchanging work centers simultaneously two at a time, beginning with a given arrangement, until no decrease in the cost is observed. As with the Wimmert technique the combinational problem lengthens the solution process as the number of work centers increases. The number of evaluations that must be made is determined by the following:

$$C_{n:r} = \frac{n!}{r!(n-r)!} \qquad (1)$$

where

$C_{n:r}$ = the number of combinations to be evaluated

n = the total number of work centers

r = the number exchanged simultaneously.

Thus as Buffa *et al*. point out, a 20 work center layout, exchanging two work centers simultaneously, would require 190 evaluations. Exchanging three work centers simultaneously increases the number of evaluations to 1140. The time requirement for the IBM 7094 computer, as used by Buffa *et al*. is trivial for an evaluation of this magnitude.

The difficulty here is that a minimum to the problem may only coincidentally have been reached. For the 20 work center case, there are n! or 20! different orderings possible. This may be simplified somewhat by symmetry with the greatest simplification occurring if the work centers can be arrayed in a square. The best that can be done with 20 work centers is a rectangle of size 4 x 5. If the number of dissimilar arrangements is represented by n!/M, where M is the measure of symmetry, and the measure of symmetry for the rectangle is 4, then the number of dissimilar arrangements for 20 work centers is reduced from more than 2×10^{18} to slightly more than 0.6×10^{18} from which, at the most, 1140 have been evaluated. Nevertheless, this technique permits a considerably larger number of evaluations in a short time, because of the use of the computer, than many of the other available techniques.

<u>Hillier's technique for nondirectional sequence demand</u>

F. S. Hillier (24) discusses a somewhat similar technique using material handling volume again as a criteria. The beginning assumption is that work centers of equal size clustered as nearly as possible to a regular rectangle can be arranged to provide a minimum volume demand. The distance factor is measured along a path connecting the central points of the work centers and moving at right angles to each other. The basic concept of the Hillier technique may be explained as follows.

Assume a layout consisting of four work centers of equal size, square in shape, and identifiable by letters A, B, C, D. Each of these work centers have a certain amount of material that must be moved to each of the other centers. For example let the work flow be as illustrated in Table 3.

Table 3. Work load flow between work centers per day

		To:			
		A	B	C	D
From:	A	-	5	1	3
	B	2	-	4	5
	C	0	10	-	5
	D	2	1	0	-

If the work centers are laid out as in Figure 2, the total daily work load flow multiplied by the distance traveled will be as shown in Table 4. (Distance is computed on a straight line distance. For example A is 2 units away from D.)

A B

C D

Figure 2. Beginning layout

Table 4. Work load - distance evaluation

	Dist.	x	Load	= L.D. Factor
A to B	1		5	5
A to C	1		1	1
A to D	2		3	6
B to A	1		2	2
B to C	2		4	8
B to D	1		5	5
C to A	1		0	0
C to B	2		10	20
C to D	1		5	5
D to A	2		2	4
D to B	1		1	1
D to C	1		0	0
		Total		57

The best layout of these work centers from a work load distance standpoint will be that arrangement which minimizes the work load-distance total. For example, if C were moved one space to the right, it would be one distance unit closer to B and one distance unit further from A. One should, therefore, subtract 1 x 10 units from the total and add 1 x 0 to the total. If D in turn were moved to the left one unit it would be one distance unit closer to A and one further away from B. Thus 1 x 2 units would be subtracted from the total and 1 x 1 units would be added to the total. The result by making this switch of C and D would be as follows:

Total = 57 - 10 + 0 - 2 + 1 = 46. Therefore, the layout has been improved.

If all such comparisons were made on a similar basis, the resulting change that produced the smallest total would be the "best" layout. However, as in the Wimmert technique and in the CRAFT technique, the number of combinations that would result as the number of work centers is increased becomes unmanageable from a practical standpoint.

As it develops, this procedure provides only a sampling from a distribution of minima that are dependent on the beginning array. Consequently, like CRAFT, one iteration from a given starting point may not provide an absolute minimum for the system. In Mr. Hillier's discussion, an example of twelve work centers is developed. One manual solution for a system of this size by a trained clerk would require about 1 1/2 hours according to Mr. Hillier. A solution for this matrix was reached when the arrangement yielded a 297 work-load volume. However, the reported efforts of 24 seniors and graduate students of Industrial Engineering at Stanford yielded an average value of 312 in an average time expenditure of 3 hours 14 minutes. One student reported a better solution, 296, after 4 1/2 hours. Unfortunately as the number of work centers increase the time requirement seems to increase at a phenomenally steep rate.

While the procedure described above illustrates the essence of the Hillier technique, some of the variations have been omitted. For example, some evaluations can be computed as the solution progresses which in turn indicate whether improvements are being made. A modified Hillier procedure has been programmed for the IBM 360 computer by the Iowa State University Numerical Analysis Office. As a result, 15-20 different starting arrangements of 12 work centers can be evaluated per minute.

"Branch and Bound" application to optimal assignment

Little *et al*. (33) developed a technique designed to solve problems of the traveling salesman type. Ideally the set of all feasible solutions is broken into increasingly small subsets by branching and computing a lower limit on the costs for each subset. Eventually a subset is found that contains a solution whose cost is less than or equal to the costs of all other lower bounds. Gavett and Plyter (15) adapted this approach to the assignment of facilities to locations. The criterion again is a function of distance and traffic intensity between facilities.

The feasibility of this technique is limited in the same way as other computational techniques. The number of branches or subsets as the number of facilities increases becomes extremely large. While ideally this approach would eventually yield the optimum results, the following summary indicates the time requirements for an IBM 7074 for an increasing number of facilities to be allocated.

Table 5. Time requirements for "Branch and Bound" facility assignment

No. Work Centers	Computing Time
4	3 sec.
5	15 sec.
6	45 sec.
7	14 min.
8	42 min.

While the authors pointed out the possibilities of improvements from refining the program or perhaps use of more rapid computers, the technique leaves something to be desired for wide use.

In general, the reported quantitative techniques designed to improve the task of facility layout have had common failings. First, the problems expand drastically in magnitude as the number of work centers increase. As a result the proposed solutions are either too time consuming to be feasible or they provide solutions that are at the lower end of the cost spectrum but not necessarily the lowest cost solution. To date, the heuristic approaches have been the most promising for feasible computer application.

Secondly, the measure of effectiveness has been, in every case, the product of distance and traffic intensity between centers. It is expected that this would be the case as these bits of information are real, measurable, and usually readily available. However, it is not the case that the distance-load factor is always of prime importance. Where other factors are crucial to the decision-maker, the various quantifications become just one of a number of factors to be considered.

COMPUTER GRID-FLOW EVALUATION OF FACILITY LAYOUT

Nature of the Procedure

The Hillier procedure, described in the previous section, has been programmed for the IBM 360. A more detailed example of the nature of the computations in this procedure is presented for a hypothetical, 6 work-center facility in Appendix A. The computer program is designed to solve a given beginning layout following this example and determine a new layout which will improve the given layout as measured by a reduced load-distance factor. This factor is determined as shown in Equation 2.

$$\text{Load-flow factor} = \sum_i \sum_j x_{ij} d_{ij} \qquad (2)$$

where x_{ij} = the total load from location i to location j

d_{ij} = number of grid-spaces, measured on the aisle, between location i and location j.

An example of the computer output for the problem evaluated manually in Appendix A is presented in Appendix B. For each evaluation made by the computer, two matrices will be shown as in Appendix B. The NN-number indicates the trial number. The top matrix indicates the starting array of work-centers arrayed schematically as they might appear in a grid overlay of the facility area to be allocated. The numbers 1-6 merely indicate a coding of an identifiable work center of a given size and specified function.

The second matrix bearing the same NN-number represents the best arrangement that can be made from the given matrix after all possible one-for-one work center trade-offs can be made. The load-distance factor is indicated as the cost and is 162 for the given solution and 129 for the improved solution.

This one improved solution is not to be considered as the optimum solution in terms of the absolute lowest cost. It is possible that an investigation of work-center trade-offs taken two, three, or more at a time would yield an even better layout. However, this would also lead to an impractical number of evaluations to be made for effective use of today's computer. This is particularly true as the number of work-centers increases.

The time requirements for computing the one-for-one trade-offs were approximately 2 seconds per NN-number, or per iteration, for the 6-12 work center case and approximately 6 seconds per iteration for a 28 work-center case.

Critical Assumptions of the Grid-Flow Layout Procedure

Several assumptions are implicit in the procedure as has been outlined for assigning work-centers.

1. The criteria of minimizing the load-distance factor is of sole significance.
2. The cost of the improved layout approaches the cost of the optimum layout.

3. All work-centers to be assigned are of equal size, dimensionally as well as in area.
4. No restrictions exist in the placement of any work-center in any location.
5. The value of the all loads between work-centers is of equal importance.
6. The units measuring all loads within the layout are the same.
7. Aisle measurements along the rectangular grid represent the actual paths over which the loads will flow.

While some of the assumptions tend to be rather restrictive, some modifications will serve to ease the restrictiveness.

Assumption 1: Adequacy of the minimum load-distance criteria There is no doubt that factors other than minimum load-distance influence layout planning and properly so. There are personnel considerations, physical limitations of plant, capital requirements, future expansion, product changes, etc., which often must be considered. This is the situation whether the layout is a new design for a facility not yet in existence or a renovation of an existing facility.

However, minimization of the flow of work will rarely be absent from the decision makers problem. Several courses are open to the designer using a grid-flow approach. One, complete the grid-flow analysis and then adjust the resulting layout according to the other factors requiring consideration. Two, arbitrarily weight the load between various work-centers which are known to be affected by other factors and proceed with the evaluation. Three, develop a desirable layout based on other criteria and subject the results to the grid-flow evaluation to test the degree of change that would be imposed by the load-distance factor.

In each case the final decision must involve subjectivity. However, the degree of subjectivity may be lessened by incorporating the results of an analysis which, because of the interrelationships, can enlighten a complex situation. At best, any layout result must be inspected to see that the logic of the layout is preserved.

Assumption 2: The improved layout vs. the optimum layout As was pointed out earlier, the improved layout is nothing more than that. However, in the computer version of the grid-flow evaluation, many different beginning layouts are subjected to the one-at-a-time interchange with the result that many different improved layouts are available for comparison. From these many improvements, that which has the lowest cost is certainly better than all of the others. There is still no assurance that this is the optimum.

The computer program is designed to randomly generate a beginning layout for each iteration. Consequently the designer has only to specify the number of iterations desired.

Since each iteration is one outcome of a population of all possible arrangements from the grid-flow evaluations, a random generation of layouts with their resulting evaluations then produces a random sample of the total population. As with any sampling, the inference about some population statistic from a sample statistic is a function of the size of the sample. The statistic of interest in this case is the lowest cost of all evaluations and consequently the larger the number of iterations the better the lowest cost layout will estimate the optimum layout.

From the Hillier data for a 12 work-center case, it was reported (24) that the best of several manual attempts was to produce a layout with a cost of 296. Of the several attempts, the modal value was a cost of 312. When these data were run on the computer, a sample of 490 iterations was generated. Figure 3 presents the result of this sample in histogram form. Approximately 16 minutes of computer time were required to generate these data with the result that the lowest value had a cost of 287 and the most frequent were in the 320-325 range.

Questions of the relative value of these must still be raised. For example, how close to the optimum is the 287, and, is it economically feasible to continue sampling to attempt to find better results, or, was the computer time justified if the intent had been solely to improve the manually

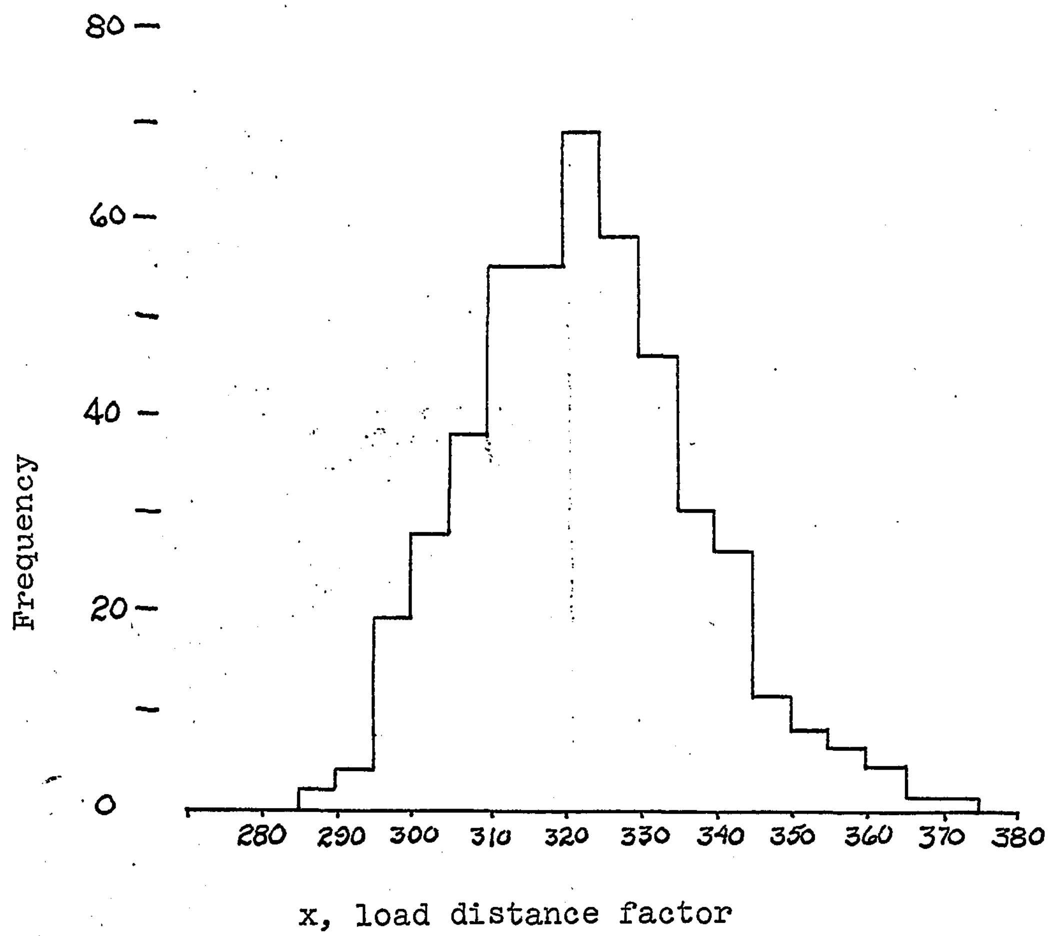

Figure 3. Histogram of the results of 490 iterations using the grid-flow procedure

developed layout with the cost of 296. These questions are again taken up in the subsequent sections of this dissertation.

Assumption 3: Work-center size Hillier suggests a feasible means of circumverting this restriction where considerable disparity exists between work centers with respect to size. The intent of the evaluation will be served if some area is considered to be a smallest common denominator and each work center is divided into multiples of this common denominator. These segments of each work center can be held together in the resulting evaluation by assigning a very high load factor in the initial load table.

This procedure was used in evaluating data provided by the U.S. Air Force for the purpose of developing a standardized base level supply layout. The results of the evaluation (37) provided a layout in which all of the work centers which had been divided to satisfy the grid-flow technique were again brought together and all other work centers oriented around these. The nature of the technique requires a precaution. Since the arbitrary high load factors between dissected centers serve to drive the segments back together very quickly, and since the procedure only exchanges work center elements one-at-a-time, the evaluation tends to become locked with the multi-element centers located early and all others forced to locate relative to those of larger mass. Thus a larger sample tends to be required where the splitting of work centers is necessary.

Assumption 4: Allocation restrictions In many instances, especially where the layout is a reallocation of existing facilities, a work center may be locked in place due to capital budgeting restrictions or other factors. The grid-flow assumes that any center is free to be located in any position and thus may shift the work center out of its fixed location in search of an optimum. An example of this requirement was present in the U.S. Air Force problem mentioned above. A computer facility was in place with certain air conditioning and special construction consistent with the requirements of the system. A new computer system was installed which would require a new supply layout for greatest efficiency in operation. Although the new computer did not require special air conditioning it was restricted to the same location as the displaced equipment. Consequently the layout had to be oriented with the computer work center fixed in a certain location. Where the fixed location is on the perimeter of the facility, a dummy row or column may added along the perimeter with the fixed work center tied to the dummy row or column by an arbitrary high load factor. An interior fixed location will require subjective reorientation of the grid-flow output.

Assumption 5: Non-priority load restrictions Where a priority exists between work centers, it will be ignored by the grid-flow evaluation. However an arbitrary weight factor may be used to increase the load factor between the priority points and then allow the grid-flow procedure to continue. An alternative, as before, is to exercise subjective orientation to the grid-flow output.

Assumption 6: Equivalent work load units   Frequently the traffic between certain centers to be located is measured in different units than traffic between other centers. This is particularly true in the case of locating service centers, e.g., supervision offices, wash rooms, etc., along with productive centers. Again one alternative is to subjectively inject these service centers into the completed layout of productive centers. Alternatively an equivalence table may be created between the different measures of traffic flow and permitting the grid-flow procedure to place all work centers simultaneously.

Assumption 7: Rectangular traffic flow   While this assumption is vital to the grid-flow procedure, the existence of an alternative situation should not invalidate the grid-flow output. Since the arrangement is a relative one, the output should approach an improved layout regardless of the travel paths actually employed. However deviation from rectangular pathways for traffic flow are rare and usually occur by virtue of overhead conveyors, pneumatic systems, etc. that use air space rather than ground area. These elements should then be designed into the resulting layout rather than dictate the initial design.

Subsequent sections of the dissertation will be addressed to the questions raised under Assumption 2. The improvements intended by these sections imply the use of the computer rather than manual evaluations and serve to make the grid-flow procedure a more complete evaluation system.

The application of the grid-flow technique in the case of the U.S. Air Force data and in several of the plant design projects prepared in the Industrial Engineering plant design courses at Iowa State University have indicated satisfactory results. In all cases, the traditional approach to the layout problem, i.e., use of templates, was used to determine a qualitative layout design. When the results of the grid-flow program, using the same data required for the qualitative evaluation, were compared, the conclusion of the persons involved was that the program layouts were at least as good as, and in several cases better than, those obtained qualitatively.

THEORY OF THE STATISTICS OF EXTREMES

Background of the Theory

According to Gumbel (20) two classifications of statistical investigation are intended to be approached by studies of extreme values. One classification seeks answers to the question, "Does an individual observation in a sample taken from a distribution, alleged to be known, fall outside what may reasonably be expected?" (Gumbel (20, p. 1)). The most common industrial application area for this class is statistical quality control.

The second class of investigation attempts to determine whether "a series of extreme values exhibits a regular behavior" (Gumbel (20, p. 1)). A more recent application area in the industrial sphere for this class of investigation has been referred to as "Life Testing" or "Reliability-Testing". Within a particular industry, the kinds of questions asked might be characterized by the following (Sarin (40, p. 2)):

"a) What is the life of the product?

b) To what extent have design, material, manufacturing process, or useage environment changes affected a product's life?

c) How effective (or costly) is the department's life warranty policy likely to be?"

The latter classification is of particular interest here as the basis for an examination of the optimization of a facility layout. A condition necessary in the analysis is that the distribution and its parameters from which the extreme values are drawn must either have a constant time or spacial relationship or they must be normalized or in some way taken into account. A further condition is that the observations from which the extremes are taken should be independent.

The question asked in (b) above addresses itself to the stability condition. In either case the questions asked in (a), (b), and (c) require attention by the engineer to subsequent questions in establishing the test rationale (Sarin (40, p. 3)):

"a) Under what environment will the tests be conducted?

b) How many objects should be tested?

c) What constitutes failure?

d) What are the consequences of wrong inferences and what are the tolerable risks associated with these inferences?

e) What precautions are to be taken to insure representivity in the objects chosen for test?"

Historically the kinds of questions indicated above have early beginnings. As early as 1709 Nichalas Bernoulli was concerned with an actuarial problem, i.e., if n men of equal age die within t years, what is the mean life of the last survivor. The practical considerations were those of insurance investments, diversification of cargo shipments, gambling decisions, etc.

Some of the first research relating to the theory of largest values was performed by astronomers who, concerned with repeated observations of the diameter of a star, were interested in establishing a basis for accepting or rejecting very large or very small values.

Gumbel (20) credits R. von Mises, subsequent to some initial work of L. von Bortkiewicz in the theory of extreme values, as having introduced in 1923 a concept of a "characteristic largest value". This characteristic largest value represented a value which the mean of the largest values drawn from a normal population approaches asymptotically. In brief, if from many samples, each arrayed with the values of the variate in ascending order, the largest value were selected, a mean largest value can be determined from the distribution of the largest values. As the size of the sample increases, this estimate of the mean of the largest values from a parent normal distribution asymptotically approaches an upper limit.

During the same year, E. L. Dodd (9) published a study which concerned itself with largest values from other distributions. This work was also based on "asymptotic" values similar to the "characteristic largest value". In 1927, according to Gumbel (20), M. Frechet published a paper based on the concept of an initial distribution different from the normal. His effort was the first to result in an asymptotic distribution of the largest value. One of his important findings showed that the largest values taken from different initial distributions having a property in common may have a common asymptotic distribution. The following "stability postulate", which was to be an important link to the development of other asymptotic distributions, was introduced in this paper.

> "If the distribution of an extreme is equal to the initial distribution except for a linear transformation of the variate, the initial distribution is called stable with respect to this extreme."
> (Gumbel (20, p. 117))

In 1928, Fisher and Tippett (14) used the stability postulate as a basis for two additional asymptotic distributions valid for initial distributions other than the normal. In 1936, Gumbel (20) reports, von Mises classified the initial distributions for which the largest values are asymptotically distributed and gave sufficient conditions for validity of the three asymptotic distributions. Necessary and sufficient conditions for the validity of these distributions were developed and presented in 1943 by Gnedenko and, as Gumbel (20) relates, proved that the three asymptotic distributions are the only ones which fulfill the stability postulate.

The application of the extreme value distributions to engineering problems have varied from flood stage predictions for flood control projects, meteorological and geological problems, to structural design problems where decisions regarding factors of safety were required. Gumbel, in 1954 (19), provided a number of numerical examples of applications.

Gumbel (20, p. 76) presents the following with regard to the mutual symmetry between the asymptotic distributions of the smallest and largest values.

> "The smallest and largest values x_1 and x_n taken from a symmetrical distribution are mutually symmetrical. If the initial distribution is asymmetrical, the symmetry principle means: From a given distribution of the largest value, valid for variate x, we may obtain a distribution of the smallest value by changing the sign of x. In two mutually symmetrical distributions the distribution of the largest value of the one is the distribution of the smallest value of the other and vice versa."

It was the assumption of symmetry (14, 32) that permitted W. Weibull (48) to use one of the three asymptotic distributions in the analysis of dynamic breaking strength of materials. From this application Weibull estimated the characteristic lowest value at failure, the minimum life, and the number of cycles before which no failure occurs. Because of this first application of what has been termed "The Third Asymptote" to analyses of breaking strengths, the distribution has more frequently been called the Weibull distribution.

Until recently, however, the asymptote which has been termed the "First Asymptote", for the exponential type of initial distribution, has been considered to be the most important. Gumbel (20, p. 246) shows that the Gompertz formula for the life table, though produced in the 1820's and thus prior to Frechet's work, is the first asymptotic probability of smallest values and was used to calculate the characteristic oldest ages at death.

However, more recent work has explored the use of the Weibull distribution for various analyses. Sarin (40, p. 16) discusses its uses in problems of reliability testing and more recently, Henderson (23) and Scigliano (42) investigated its application in the area of service life estimation for industrial property. It will be shown that this distribution permits a practical solution to the problem of facility layout optimization in general.

Distribution of Extremes

The first studies assumed the initial distribution of the variables to be normal. It was found that the analytical results were complex and that much could be derived by first assuming an exponential distribution. It is the purpose of this section to summarize the exact distributions of the extremes as functions of sample size and the properties of the initial distribution. Subsequent sections will summarize the three asymptotic distributions as the sample size, n, becomes very large.

If a continuous variate (x) having a cumulative distribution function F(x) is assumed, then the probability that a sample of n independent observations is made in which all n observations are less than X is as follows:

$$P(x_1, x_2, x_3, \ldots x_n \leq X) = [F(X)]^n \tag{3}$$

However, for simplification of notation, let

$$\begin{aligned} \phi_n(x_L \leq X) &= p(x_1, x_2, x_3, \ldots x_n < X) \\ &= [F(X)]^n \end{aligned} \tag{4}$$

From this relationship, the probability that X is the largest value becomes smaller as the sample size, n, increases. For different values of X, the graph of $\phi_n(x_L \leq X)$ plotted against n will form a family of non intersecting curves which will shift to the right with increasing values of X. Figure 4 illustrates this for an initial Poisson distribution with a mean of 5.

By symmetry, the probability that all of n independent observations are greater than X may be written

$$p(x_1, x_2, x_3, x_4, \ldots, x_n \geq X) = [1 - F(X)]^n \tag{5}$$

or that the probability of the smallest among n independent observations is less than x, denoted by $\phi_n(x_s \leq X)$ is

$$\begin{aligned} \phi_n(x_s \leq X) &= 1 - p(x_1, x_2, x_3, x_4, \ldots x_n > X) \\ &= 1 - [1 - F(X)]^n \end{aligned} \tag{6}$$

Figure 5 illustrates this relationship again using a Poisson parent distribution with a mean of 5. For the smallest values, the curves shift to right with decreasing values of X while the probability of the smallest sample value

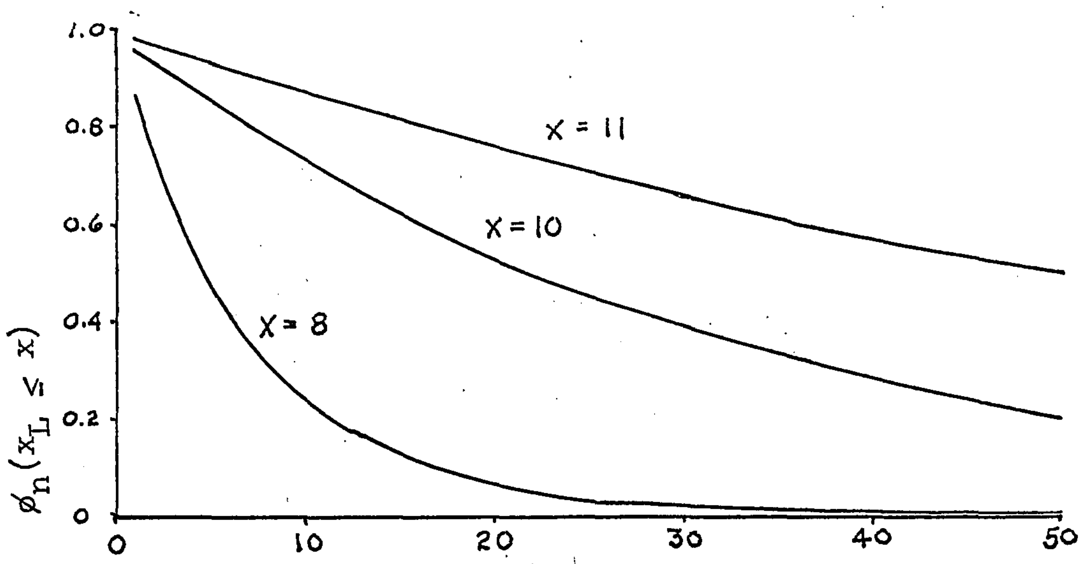

Figure 4. Largest value probabilities for a Poisson distribution with mean of 5

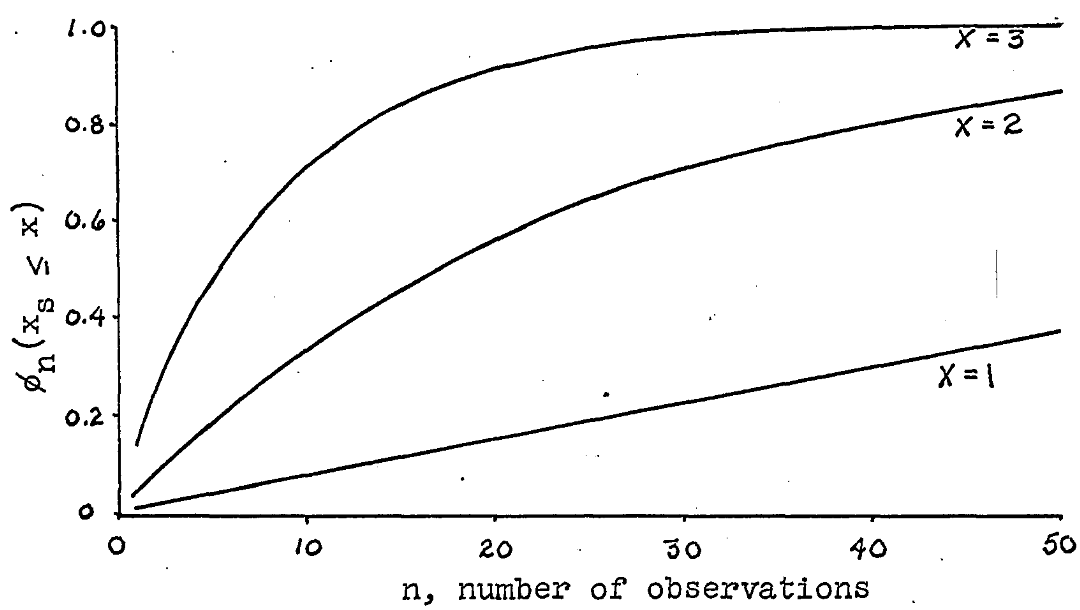

Figure 5. Smallest value probabilities for a Poisson distribution with mean of 5

being less than x increases with an increase in n. From these distributions, the density functions for the largest and smallest values, denoted by $p_n(x_L \leq X)$ and $p_n(x_s < X)$ respectively, are found by taking the derivative with respect to X or

$$p_n(x_L \leq X) = \frac{d[\emptyset_n(x_L \leq X)]}{d(x)} = n[F(x)]^{n-1}F(x) \qquad (7)$$

$$p_n(x_s \leq X) = \frac{d[\emptyset_n(x_s \leq X)]}{d(x)} = n[1-F(x)]^{n-1}F(x) \qquad (8)$$

Since the function of particular interest to the identification of the optimum facility layout is the one which would permit some predictions from the distribution of smallest load-distance factors, further discussion will be limited to the distribution of smallest values. For life estimates, the initial distribution was of the ages at death, or the largest values. For flood prediction, the extreme of interest was the peak and so consequently the distribution of the highest stage attained each year was the initial distribution of interest. Reliability studies are life studies in which the service life of the component is the statistic of interest and in effect are life estimate studies of inanimate objects. Weibull's work was concerned with the loadings at rupture of various metals and in which the rupture loadings formed the initial distribution.

Significant Statistics of Extreme Distributions

From Gumbel (20, p. 79) the quantiles of the extreme value distributions may be estimated as follows:

$$F(q_n) = F({}_2x_{kn}) = e^{\frac{-k(\ln 100/q)}{n}} \qquad (9)$$

$$q_n = {}_2x_{kn} \qquad (10)$$

$$1/k = 6.63483 - 1.44290(\ln q) \qquad (11)$$

where ${}_2x_n$ = the median of the extreme values in a sample of size n.

q_n = the quantile of the sample of size n.

k = the sample size multiplier indicating the number of observations for which q_n would be the median.

An additional statistic introduced by Gumbel is the <u>characteristic smallest value</u> (X = v) (Gumbel (20, p. 82)). This is defined as

$$n\,F(v) = 1 \qquad (n \geq 2) \qquad (12)$$

such that

$$\emptyset_n(x_s \leq v) - 1 - \frac{1}{e} \qquad (13)$$

or if N smallest values are taken from N samples, each of size n, approximately 63.2% will be below the characteristic smallest value. This value is significant for the interpretation of the intersection of two consecutive distribution functions, i.e., $p_n(x_s \leq X) = p_{n+1}(x_s \leq X)$. This relating principle is stated by Gumbel as follows:

> The distribution of the smallest among n observations intersects the preceeding distribution of the smallest among n+1 observations at the characteristic smallest value of the latter distribution.

Figure 6 illustrates this principle for a Poisson distribution with mean of 5. The intersection of the curve $n = 1$ with $n+1 = 2$ is approximately 5.3 and of $n = 2$ with $n+1 = 3$ is approximately 4.2. These correspond to the values of v from Equation 12 and cumulative probability tables for $n = 2$ and 3 respectively.

Gumbel describes three initial distributions of extreme values which are generally classed as exponential distributions. The first is identified as the exponential type, a second is described by the Cauchy and Pareto distributions, and third which is described only as being limited either to the right or to the left with respect to the initial variates. For each of these initial distribution classes an asymptotic distribution, or asymptote, exists for which only the parameters depend on the initial distribution.

The Exponential Type and the First Asymptote

The characteristics of the exponential type of distribution include an unlimited variate in the upper or lower extremes, the existence of all moments, and the relationship

$$\lim_{x \to \infty} \frac{f(x)}{1-F(x)} = - \frac{d \ln f(x)}{dx} \qquad (14)$$

These conditions are necessary but not sufficient for the

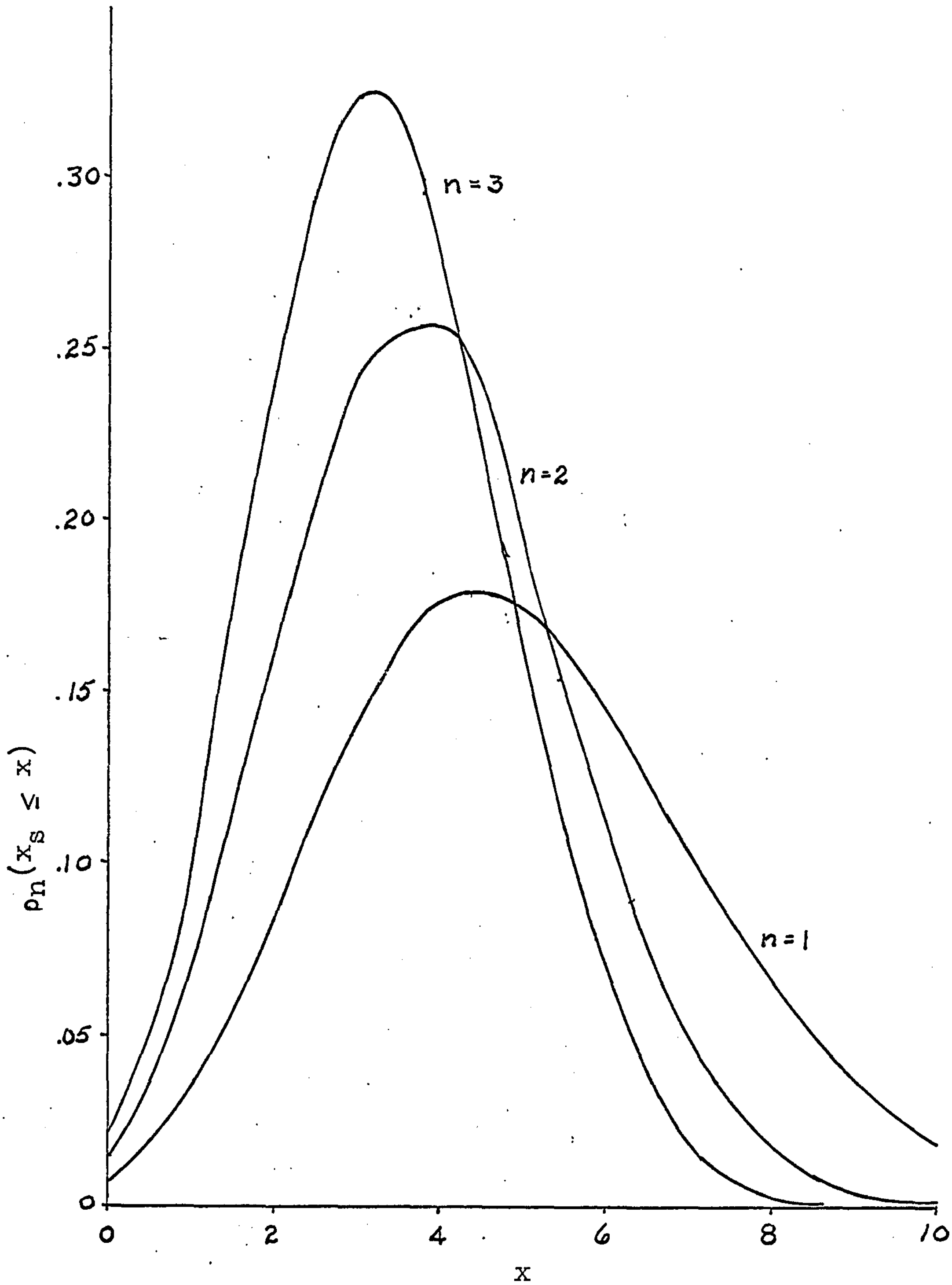

Figure 6. Intersection of extreme value distribution at the characteristic value

exponential type of distribution. In addition, a variate limited at its lowest value may be of the exponential type for the highest values if these are unlimited, and conversely. For this distribution the mode converges toward the characteristic value, the median of the smallest value is smaller than the mode, and the distribution of the smallest value is negatively skewed. In addition the occurrence interval converges to the sample size, n.

An asymptotic distribution, termed the first asymptote at $n = \infty$, associated with the exponential type for the smallest values is

$$F(x) = e^{-e^{\alpha(x-u)}} \tag{15}$$

$$f(x) = \alpha e^{\alpha(x-u)-e^{\alpha(x-u)}} \tag{16}$$

where α, the extremal intensity function, is a parameter associated with the shape of the distribution and u is associated with the characteristic value. Both of these parameters are dependent on the initial distribution. The estimation of these parameters may be obtained from the initial distribution if it is known and if the sample size n is known. In general, however, these facts are not known. For example in the flood crest or life estimating problems, only the largest values are known or observed. As a result the parameters may be estimated based on order statistics.

The Pareto and Cauchy Distributions and the Second Asymptote

In general, the parent distributions identified by this second classification have no moments or only a finite number of moments exist. This characteristic also holds for the distribution of extreme values. For these distributions, the mode increases more rapidly with n than the exponential type but does not converge to the characteristic value. Also the occurence interval does not converge to n. Examples of these distributions are given by the Pareto distribution which leads to a type of distribution limited as follows:

$$\lim_{z=\infty} z^k[1 - G(z)] = A > 0 \qquad (17)$$

$$k > 0; \quad z \geqq 0$$

where k is a shape parameter of the distribution of the variate z. The Cauchy distribution leads to a type of distribution which may be said to belong to the Pareto type in both directions. This type has, in addition to the characteristic represented by Equation 17, a characteristic as follows:

$$\lim_{z=-\infty} [G(z)][-z]^{k_1} = A_1 > 0 \qquad (18)$$

$$k_1 > 0 \quad z \leqq 0$$

If the initial distribution is symmetrical, $k_1 = k$, and $A_1 = A$. Thus the Pareto type refers to distributions which are unlimited with respect to the variate of interest either at the upper or lower values and the Cauchy type refers to those which are unlimited in both directions. These are not sufficient conditions however because distributions exist which are unlimited at either extreme but which do not belong to either the Pareto-Cauchy, or exponential types.

The asymptotic distribution, termed the second asymptote, of the lowest values for initial distributions of the Pareto-Cauchy types are stated in Equations 19 and 20. The second asymptote may be obtained by identifying a class of initial distributions satisfying certain requirements, by identifying asymptotic properties of certain initial distributions and calculating the asymptotic distribution, or from the first asymptote by a logrithmic transformation.

$$G(x) = e^{-\left[\frac{w-v}{w-x}\right]^{k}} \tag{19}$$

$$g(x) = \frac{k}{w-v}\left(\frac{w-v}{w-x}\right)^{k+1} e^{-\left[\frac{w-v}{w-x}\right]^{k}} \tag{20}$$

where w = an estimate of an upper limit if one exists

v = the characteristic value

k = a dimensionless inverse measure of dispersion, analogous to α in the first asymptote.

The estimation of these three parameters may be accomplished by a maximum likelihood solution. Kao (26) discusses several means of estimating the parameters.

Limited Distributions and the Third Asymptote

The third type of distribution is concerned with variates which are limited either to the right or to the left. The third asymptote may be obtained from the first by a logrithmic transformation (See Appendix D) and, like the first, possesses all moments. In studying one extreme, no assumption need be made regarding the initial distributions behavior at the other extreme. The third asymptotic probability and corresponding density function are shown in Equations 21 and 22 below.

$$H(x) = e^{-\left[\frac{x-\epsilon}{v-\epsilon}\right]^{k}} \qquad (21)$$

$$\eta(x) = \frac{k}{v-\epsilon} \left(\frac{x-\epsilon}{v-\epsilon}\right)^{k-1} e^{-\left[\frac{x-\epsilon}{v-\epsilon}\right]^{k}} \qquad (22)$$

subject to the conditions

$$v > \epsilon; \quad k > 0; \quad H(\epsilon) = 1; \quad H(v) = 1/3 \qquad (23)$$

where ϵ = the lower limit of the variate

v = the characteristic value

k = a dimensionless inverse measure of dispersion.

Other statistics of the third asymptote which are of interest are the mean ($\bar{x}$), the median ($_1x$), the mode, ($_2x$) the reduced moments about the origin, and the variance. These may be represented by the Equations 24-28.

$$\bar{x} = \epsilon + (v - \epsilon)\ K\ (1+1/k) \qquad (24)$$

$$_1x = \epsilon + (v - \epsilon)(\lg.2)^{1/k} \qquad (25)$$

$$ {}_2x = \epsilon + (v - \epsilon)(1-1/k)^{1/k} \tag{26} $$

$$ \overline{(x-\epsilon)^{\ell}} = (v-\epsilon)^{\ell}\, K(1+\ell/k) \tag{27} $$

$$ \sigma^2 = (v-\epsilon)^2 [K(1+2/k) - K^2(1+1/k)] \tag{28} $$

Based on these functions, a relationship exists between the median and the mode depending on the parameter k. Thus the mode precedes, equals, or exceeds the median depending on whether k is less than, equal to, or greater than 3.25889.

Estimation of the Parameters for the Third Asymptote

In general, three techniques are useful in estimating the three parameters ϵ, v, and k of the third asymptote. As with the first asymptote, the initial distribution and the sample size is frequently unknown. It is normally the situation that certain data have been observed and a priori knowledge indicates that the initial distribution is limited and that it may be subject to Gnedenko's necessary and sufficient condition for the existence of the smallest value (Gumbel (20, p. 163)). Consequently the parameters may be estimated by sample values from N observed minimum values, from order statistics, or graphically.

The skewness of the third asymptote as measured by $\overline{(x-\epsilon)^3}/\sigma^3$ is dependent only on the parameter k. As a result, use of the computed sample skewness may be used to estimate k. Using the standardized differences of the variate between the characteristic value and the lower limit, Equation 29, the standardized difference between the characteristic value and the mean, Equation 30, the obtained estimate of k, the sample estimate of the mean, $\bar{x}_0$, and standard deviation, s, the parameters v and ϵ may be estimated from Equations 31 and 32.

$$\frac{v-\epsilon}{\sigma} = [K(1+2/k) - K^2(1+1/k)]^{-1/2} \tag{29}$$

$$\frac{v-x}{\sigma} = [1-K(1+1/k)][\frac{v-\epsilon}{\sigma}] \tag{30}$$

$$v = x_0 + s\ [\frac{v-x}{\sigma}] \tag{31}$$

$$\epsilon = v - s\ [\frac{v-\epsilon}{\sigma}] \tag{32}$$

These procedures are simplified further by obtaining an estimate of the characteristic value, v, by order statistics. By counting the sample values from the lowest to the highest, the characteristic value, v, is assumed to be the m-th value. From previous discussion, approximately 63.2% of the values will be less than the characteristic value. Hence

$$m' = 0.632(N+1) \qquad m < m' < m+1 \tag{33}$$

Estimating v this way reduces the computation of k if tabular values of $\frac{v-\bar{x}_0}{\sigma}$ and $\frac{v-\epsilon}{\sigma}$ for decreasing values of k are available.

The graphic solution for the parameters employs the use of a lnln plot on "Weibull" paper, a sample of which appears in Appendix F. From Equation 21 the logarithmic transformation is as follows:

$$-\ln H(x) = \ln \frac{1}{H(x)} = \left[\frac{x-\epsilon}{v-\epsilon}\right]^k \quad (34)$$

$$\ln \ln [H(x)]^{-1} = k[\ln(x-\epsilon)]-k[\ln(v-\epsilon)] \quad (35)$$

$$\ln(x-\epsilon) = \ln(v-\epsilon) + 1/k(\ln\ln[H(x)]^{-1}) \quad (36)$$

This general equation for a straight line describes the curve that would result on "Weibull" probability paper if the log of the variate adjusted by a minimum value is plotted against the lnln of the cumulative probability of the sample values equal to or greater than the variate. It should be noted that the standard "Weibull" paper permits a plot of the cumulative probability of sample values equal to or less than the variate by the following transformation:

$$\ln(x-\epsilon) = \ln(v-\epsilon) + \frac{1}{k} (\ln\ln[1-H(x)]^{-1}) \quad (37)$$

By assuming various values for ϵ the reduced variate, $(x-\epsilon)$, may be plotted with the resulting best straight line fit indicating the proper value for ϵ. The slope of the straight line is $1/k$, and the characteristic value, v, may be computed from the graphic values or by any of the other methods discussed above.

Using the form from Equation 37, i.e.,

$$1 - H(x) = 1 - e^{-A^k} \quad (38)$$

where $A = \frac{x-\epsilon}{v-\epsilon}$ (39)

the change in the values of the probabilities as A and k vary may be noted in Figure 7. A tabular presentation of these

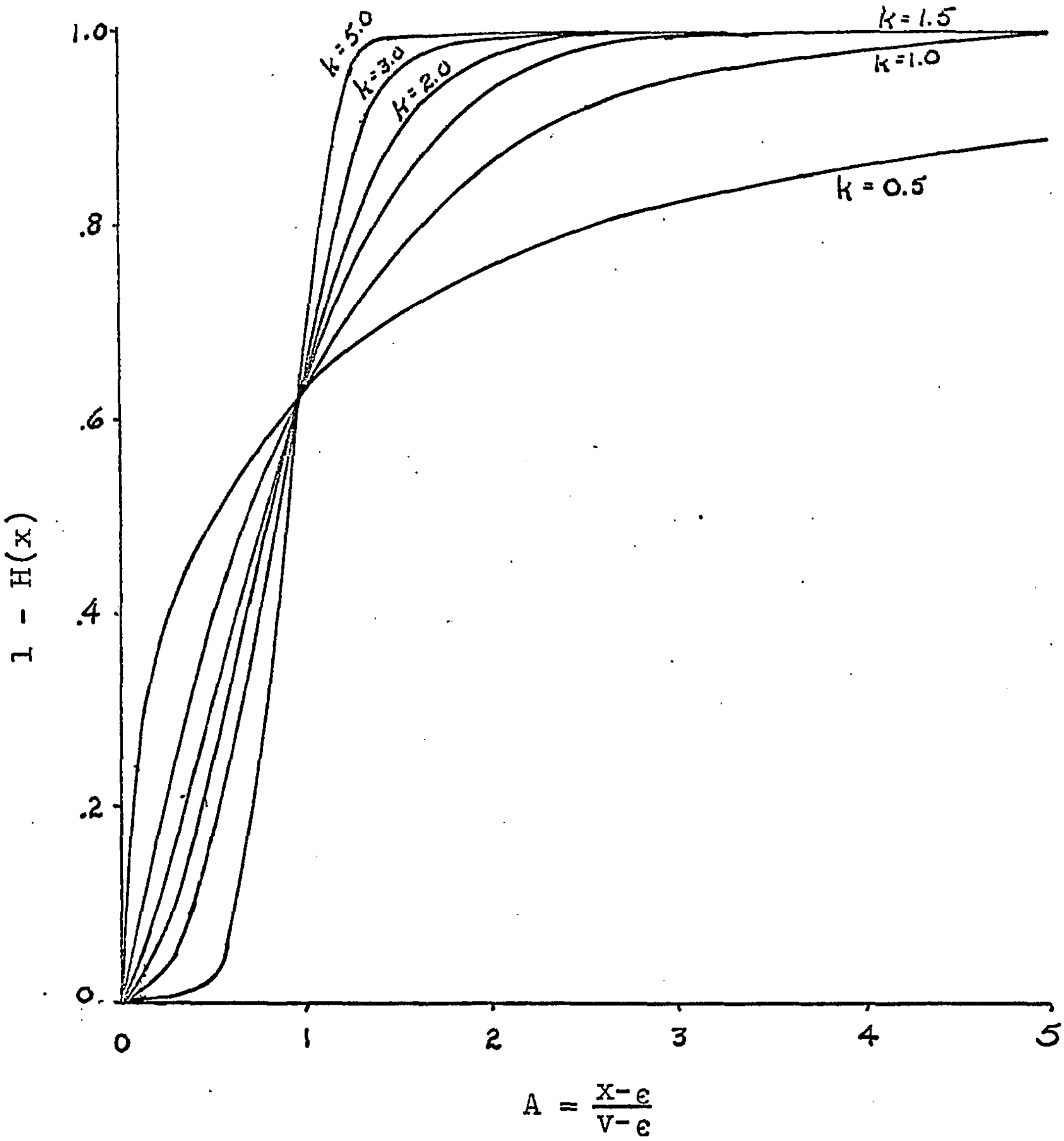

Figure 7. Change in probabilities with varying A and K

probabilities appears in Appendix C, Table 15.

Other means of estimating the paremeters are presented by Kao (26) and Lieblein (29, 30, 31).

APPLICATION OF THE THIRD ASYMPTOTIC DISTRIBUTION TO THE GRID-FLOW ALGORITHM

Measures of Effectiveness

The answers provided to the layout planner by the grid-flow program are useful and important to the better design of a facility. The basic information usually gathered by the planner in the form of product or material flows is identical to the information necessary for the grid-flow analysis. Consequently the time consuming trial-and-error arrangement of scale models or of manual manipulation of such computational schemes as cross-charting may be replaced by the rapid, computer developed layouts. In addition, the planner may be presented with a number of alternative arrangements with a measure of the degree of improvement by one over another.

However, one improvement typically results in requests for more information and the fulfillment of an insatiable quest for better bases for decision making. Following the presentation of a grid-flow layout the planner now can ask questions such as the following.

1. Since the load distance factor cannot be reduced to zero without abolishing the entire production system, what is the lowest factor that can be attained with the specialized work centers?

2. If there is no feasible way in this system to deterministically arrive at the optimum layout, what is the probability of achieving a better layout than the best one now available?

These questions are significant for the reason that if the probability of getting a better answer can be estimated, and the absolute value of the optimum layout can also be estimated, the necessary ingredients are available to determine the economic feasibility of searching for a better solution. One way of presenting this concept is by arbitrarily establishing a two-way dichotomy as in Figure 8.

Optimum minus Best Attained Layout \ $p(x_L \leq X)$	High	Low
Small Diff	Questionable Feasibility	Not Feasible
Large Diff	Highly Feasible	Questionable Feasibility

Figure 8. Economic feasibility of continuing the grid-flow search for a better layout

With this purely conceptual model it is easy to categorize the economic feasibility of continuing the search for a better layout by computer or stopping with the present estimate. The actual determination of such feasibility however must be a function not only of the probability but also of the costs of continuing the search and the cost savings for each load-factor unit that potentially can be deducted. Since the latter values of costs and savings are externally determined, the concern here is to develop the means of estimating the probability.

Problem Characteristics

Output data from the grid-flow program represents a series of iterations, or a series of samples, from the population of all reduced layout arrangement combinations possible. Each iteration is started from a randomly selected arrangement of work centers. From the starting array, all exchanges of work centers, taken one at a time, are made in an orderly fashion and the exchange which results in the lowest cost is retained. Thus, in effect, for a given number of work centers there is a series of samples each of size n. The size of the sample, n, is dependent on the number of work centers which dictates the number of single work center exchanges that can be made. Since the lowest extreme of the sample is the statistic of interest, then the second class of statistical investigation identified by Gumbel (20) is indicated, i.e., does the series of lowest grid-flow values exhibit a regular behavior.

One condition necessary, that of constant time and spacial relationship of the parent distribution of all possible layout arrangements from a given set of work centers, is unquestionably met. The characteristics of the work centers, the number of centers, the size of each work center, etc. all remain the same for any given evaluation. The initial data of traffic intensity between specified work centers remains the same throughout the analysis.

Another condition necessary, that of independence of the observations which constitute the sample from which the extremes are taken, is subject to question. Each iteration with its given starting array has a unique minimum value. The same starting array, by nature of the procedure, will always conclude with the same minimum cost array. Similarly each element of its sample will always be the same. Consequently the observations are not independent but are very much dependent on the beginning array.

Two assumptions are made regarding the point of observation independence which permit continuing consideration of the distribution of extremes for this problem. The first is that all starting arrays within the finite number of samples, if randomly generated, are independent and thus all extreme values generated are random and independent. The large number of possible arrangements in the population from which the starting array is randomly chosen makes this assumption feasible. An earlier section used the illustration of 9.6×10^{18} different arrangements in the population of all arrangements of 20 work centers, excluding the effects of symmetry.

The second assumption is that the interdependence within each sample does not significantly influence the outcome. The basis for this assumption is taken from Gumbel (20), Watson (47), and Gurland (21). Gumbel contends (20, p. 164):

> "For the asymptotic distribution of extremes, the initial distribution to be used for interdependent observations is very complicated. However, the distribution of extreme values depends only on the properties of the initial distribution for large values of the variate where the influence of interdependence may vanish. Therefore, the asymptotic distribution of extremes may still be valid for interdependent observations."

> "A sequence of variable x_i is called m-dependent if $|i-j| > m$ implies that x_i and x_j are independent. If the variables have a finite upper bound, the largest among n observations tends with probability one to this bound. Watson shows that if the variables are unlimited, the asymptotic distribution of the largest value is the same as in the case of independence."

> "The Gamma distribution is of the exponential type. Consequently, the distribution of its largest value converges to the first asymptote. Since the mean of a Gamma distribution is again subject to a Gamma distribution, the first asymptotic distribution holds for the largest means, provided the observations are independent. However, Gurland has shown that this remains valid for the largest means of uncorrelated and, what is more, for positively correlated observations taken from a multidimensional Gamma distribution. Again the independence is less important for the theory of extreme values than it seemed at first sight."

And finally this application will definitely be bounded at both extremes. As long as the initial load table contains values greater than zero there will be a positive lower limit greater than zero. It is also true that the layout arrangements will always have a finite upper bound. However, this statistic is of no concern to this problem and the extreme value distribution permits considerations of one extreme of interest without requiring an assumption of the behavior of the opposite extreme.

Application to the Hillier Data

Figure 3 illustrated the distribution estimate of the grid-flow minimum values for 490 iterations based on the data presented in the Hillier (24) article. By observation, it does not appear to be skewed significantly either to the right or to the left. If the hypothesis of the applicability of the extreme value distribution is correct, the k value for Equation 19 should approach the symmetry value of 3.26.

The Weibull distribution, or the third asymptotic distribution, was chosen because one of the parameters, ε, is the estimate of the lower bound, or optimum cost, and consequently of considerable interest.

First estimates of the parameters of Weibull were made by use of the Weibull probability paper. Equation 40 represents the form used to plot the data and is merely a simplification of the logarithmic form of the general Weibull represented in Equation 37.

$$\ln (x-\varepsilon) = \ln (v-\varepsilon) + \frac{1}{k} (y) \qquad (40)$$

By accumulating the data and determining the cumulative per cent that was equal to or less than increasing values of x from the 490 extreme values was computed. Figure 9 shows

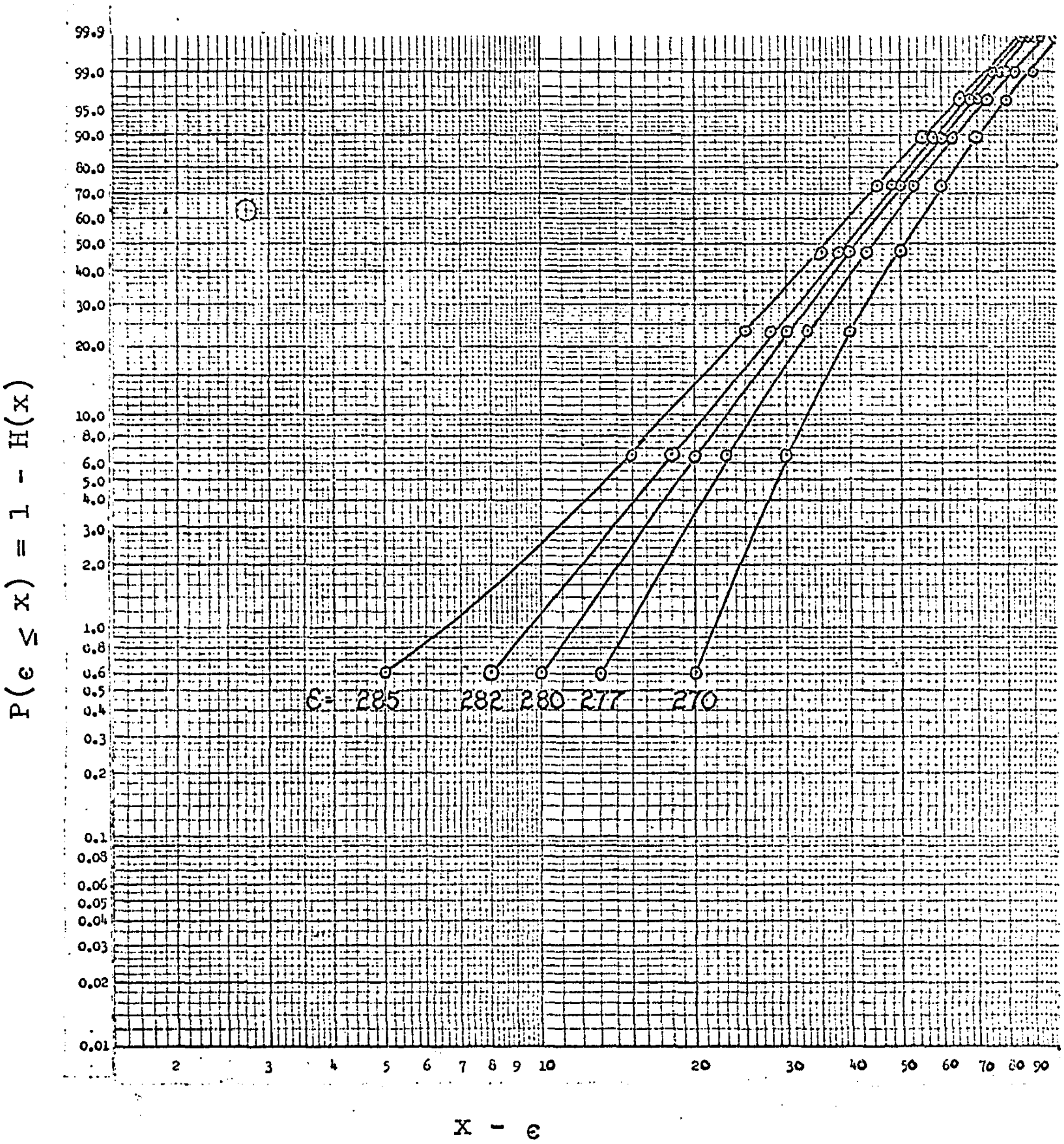

Figure 9. Weibull curves plotted from the Hillier data

the results of three plots at estimates for ϵ of 285, 280, 277, and 270 respectively. From this display, the 280 curve appears slightly convex and the 285 appears slightly concave. Since the lower ranges of the curve should be more indicative due to the cumulative error effect of estimating the probabilities cumulatively from 0 to 1, it would appear that the best straight line fit would occur for a value of ϵ between 280 and 285. A line for the value ϵ = 282 is also shown on Figure 10. Table 6 shows the estimates of the slopes of the best fit straight lines and the parameter v for several initial estimates of ϵ.

Table 6. Graphic estimates of Weibull parameters for the Hillier data

Curve/Parameter	ϵ	v	k
277	277	330	3.48
280	280	321	3.08
282	282	321	2.98
285	285	328	2.42

A second method was employed to obtain estimates for this distribution that was considerably more precise. Following the graphic presentation of the data and the resulting estimates, a linear regression model was used to determine which set of parameters would yield a regression line with a minimum sum of squares between the estimated and observed Weibull plot. The parameters for curve 282 were indicated as the appropriate estimates for fitting the observed data.

A third method employed was developed by Hartley (22) and programmed by Atkinson (2) for the IBM 7074 and 360/40 computers. This technique permits the estimation of the parameters of a non-linear, differentiable function directly requiring only an initial estimate of the parameters in the neighborhood of the expected values. Table 7 shows the TARSIER computed estimates of the parameters.

Table 7. TARSIER computed estimates of Weibull parameters for the Hillier data

ε	282
v	326
k	2.96
Sum of Squares	.003106

Figure 10 shows the plot of observed data points, the cumulative distribution determined by the initial estimates input into the TARSIER program, and the cumulative distribution determined by the TARSIER program estimates. The divergence at the upper extreme would likely be due to the effect of cumulative error in the cumulative probability estimates from the observed data.

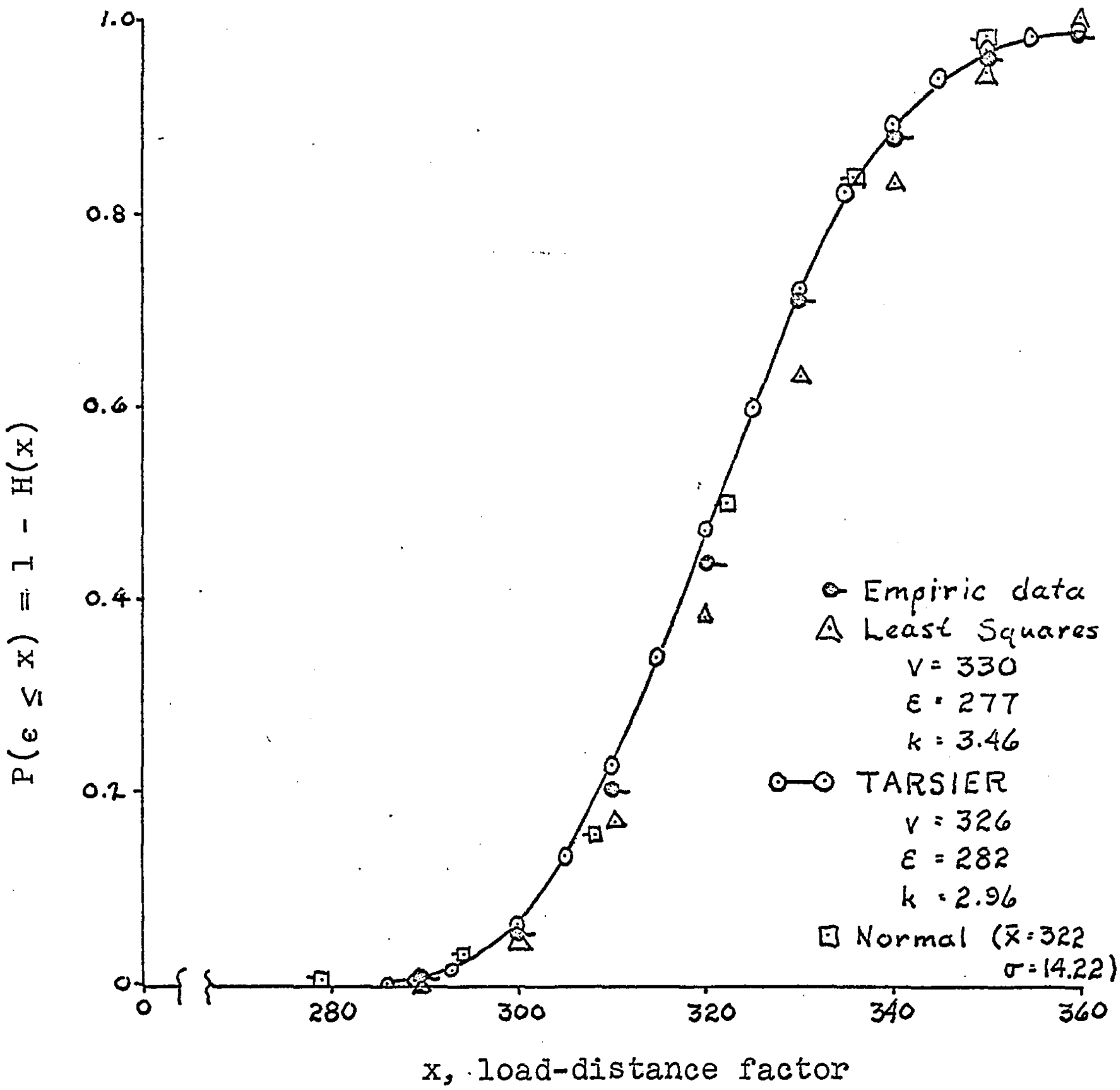

Figure 10. Observed and computed Weibull probabilities from the Hillier data

Since this data approaches a symmetrical distribution, a normal distribution based on estimates obtained from the observed data has also been plotted on Figure 10. While this also fits the data very well it is limited to symmetrical distributions of extreme values and of course does not provide an estimate of the lower bound.

Application to USAF Data

Data supplied by the U.S. Air Force represented traffic intensity in terms of document flow. Input to the grid-flow procedure were data for 15 work centers, eight of which were split according to area, resulting in a total of 28 grid elements. Elements which were to be located adjacent to each other to comprise a total work center were loaded very heavily in the initial load table. For example, the data presented represented documents per month with load magnitude as much as 300,000. Those elements to be locked together were loaded with values of 10^6 which served to keep the areas together very well. The cost values in this case are coded with the first three digits of the cost used in the extreme value analysis. A total of 25 iterations of these data were run on the computer with the result that the distribution of extremes had the appearance of being positively skewed.

The Weibull plot of the aggregated cumulative probability is shown in Figure 11 for estimates of ϵ at 320, 325, 330, and

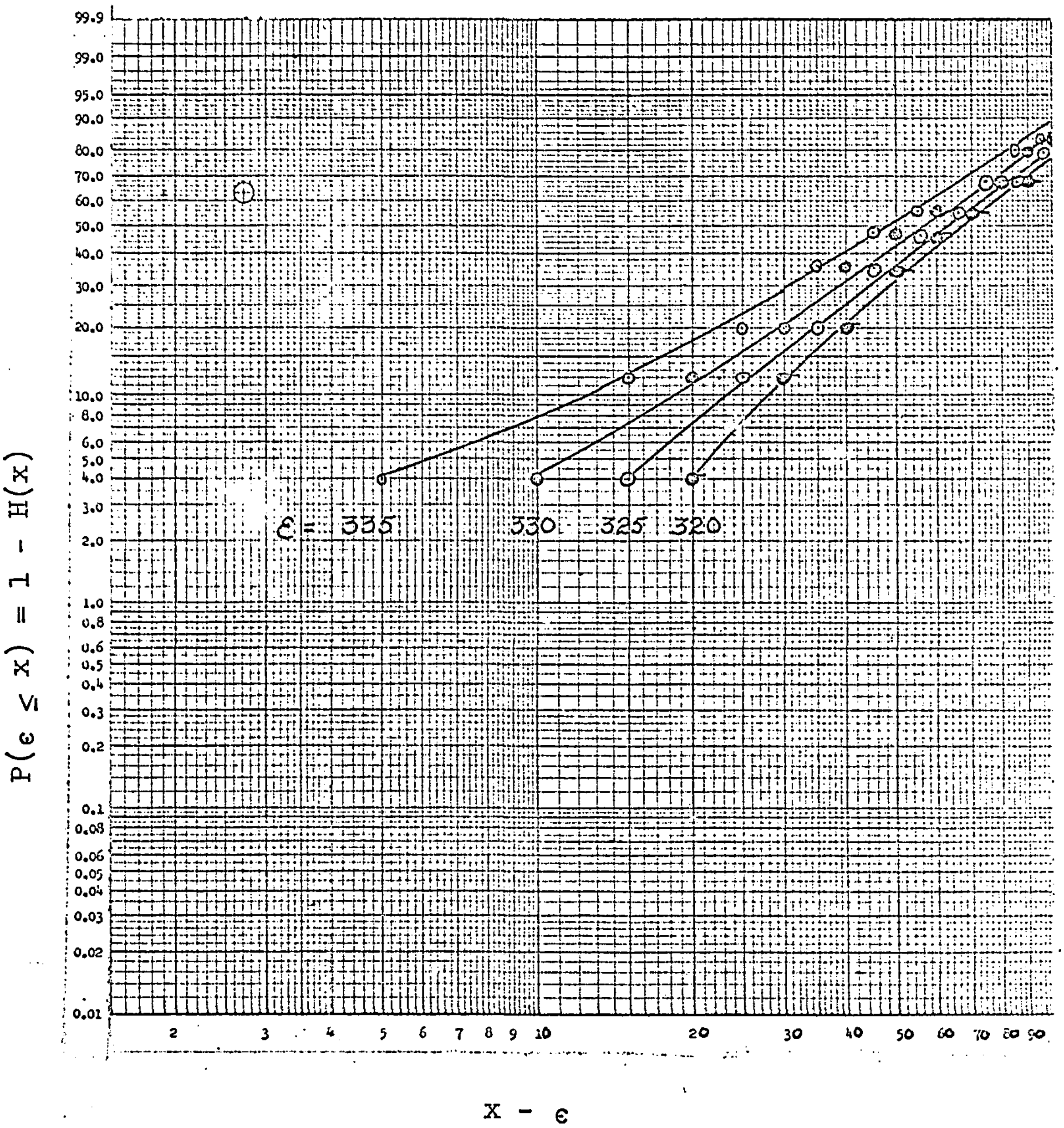

Figure 11. Weibull curves plotted from the USAF data

335. Table 8 shows the graphic, least squares and TARSIER estimates of these data.

Table 8. Weibull distribution parameter estimates for USAF data

Curve/Parameter	ϵ	v	k	Sum of squares
320	320	398	1.96	.06
325	325	398	2.36	.06
330	330	397	1.70	.05
335	335	395	1.39	.1
TARSIER	325	398	1.76	.0183

Figure 12 shows the plot of the observed data, the distribution determined by the TARSIER input estimates, and the distribution determined by the TARSIER output estimates. The Weibull distribution plotted with the normal distribution, based on parameters estimated from the data, is shown in Figure 13.

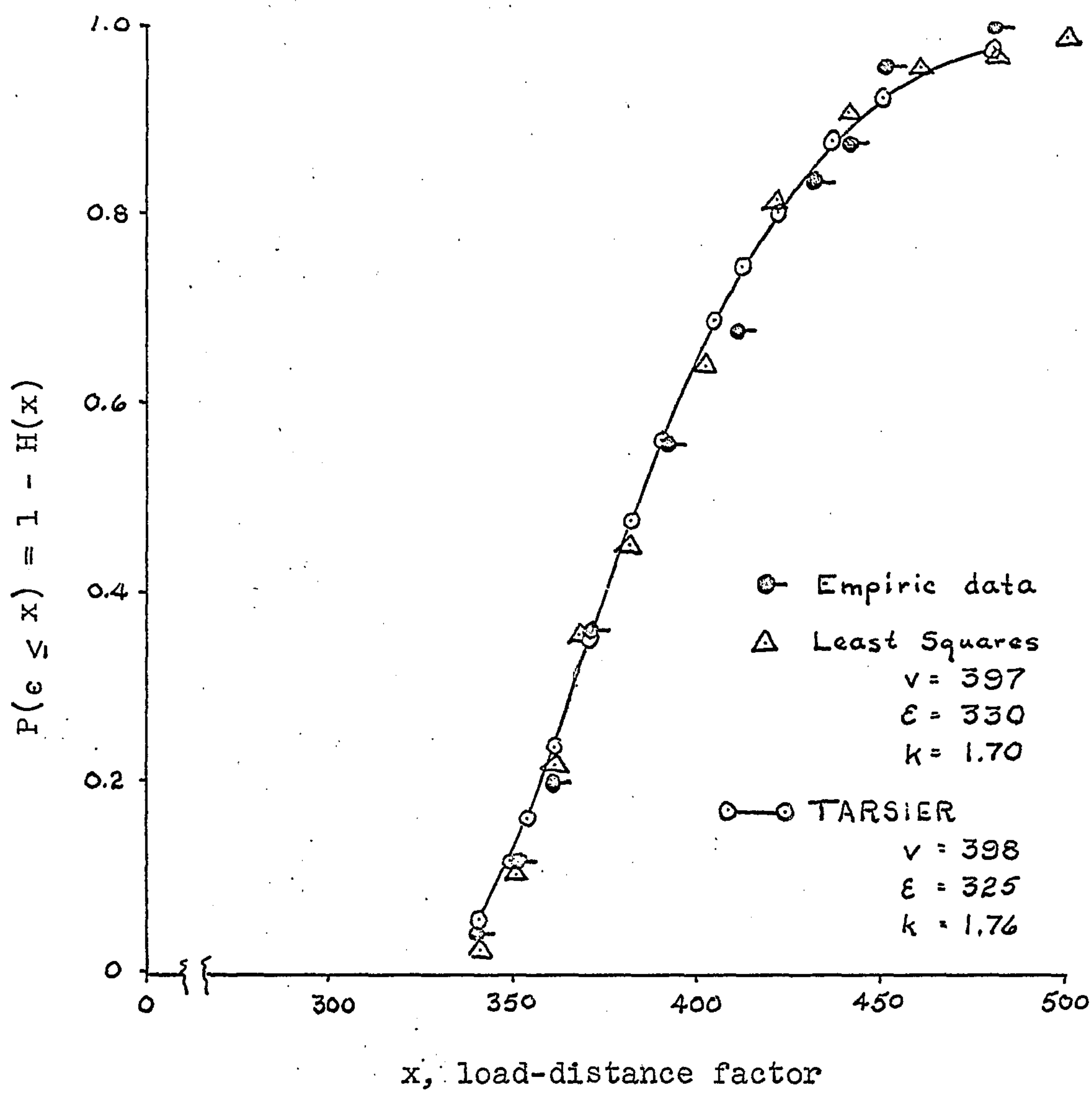

Figure 12. Observed and computed Weibull probabilities from the USAF data

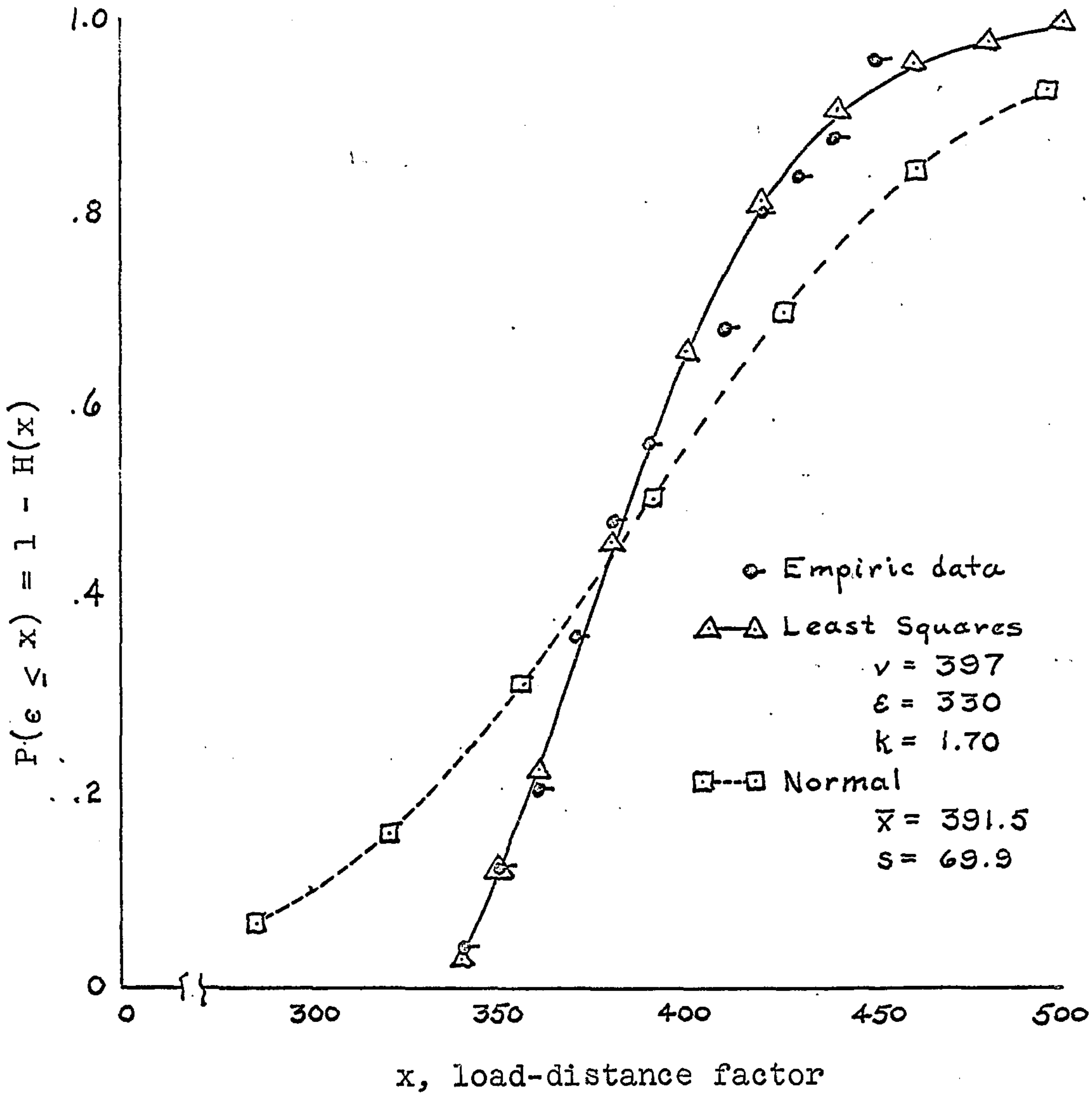

Figure 13. Weibull and normal probabilities computed from the USAF data.

RESULTS AND CONCLUSIONS

The Hillier procedure, Buffa's CRAFT system, and others, approach the problem of layout design heuristically. However, the previous discussion indicates that these heuristic outcomes can be made considerably more valuable by providing a measure of efficiency. With the Hillier data, it can now be estimated that the optimum layout under a traffic intensity criterion will have cost with a lower bound of 282. It can also be estimated that the probability of improving the present best estimate of 286 by continued trials is less than 0.1%. For the USAF supply system layout the lower bound, estimated at 340, has a slightly better than 6% chance of being bettered by an additional trial. These probabilities can be computed or determined from a table such as in Appendix C, Table 15.

This information provides the facility planner with a basis for looking further for the optimum or deciding that it would be uneconomical to do so. This decision rule is based on the probabilities for improvement and the known absolute improvement potential determined by the application of the extreme value distribution. Assuming that the economic cost for making additional trials and the opportunity cost resulting from a reduction in the measure of layout cost can be estimated, the decision rule may be qualitatively stated as follows.

1. If the cost of additional trials is greater than the expected opportunity cost of improving the layout, then the best layout generated should be accepted as "good enough".

2. If the cost of additional trials is equal to or less than the expected opportunity cost of improving the layout, then an additional trial should be made.

Quantitatively the basic equation is represented by Equation 41

$$C_c - C_F \{ \sum_{L}^{N} [\rho(x_i - x_{i-1})][x_i - x_{i-1}]\} \gtreqless 0 \tag{41}$$

where

$\rho(x_i - x_{i-1})$ = incremental change in the probability

$[x_i - x_{i-1}]$ = incremental change in the layout value

C_F = opportunity cost for each unit of layout value

C_c = computer and other costs per trial associated with generating additional trials

L = optimum layout cost

N = lowest observed layout cost.

If $(x_i - x_{i-1})$ is unity then the summation value will be the cumulative value of the probabilities of layout costs lower than the best observed. Equation 41 then simplifies to Equation 42.

$$\frac{C_F}{C_c} [1 - H(x_n)] \lesseqgtr 1 \tag{42}$$

From Equation 42, the decision rule for the facility planner may now be restated as follows:

1. If $\frac{C_F}{C_c}[1 - H(x_n)] < 1$, then accept the best layout generated so far.
2. If $\frac{C_F}{C_c}[1 - H(x_n)] \geq 1$, then make an additional trial.

Using these equations, the threshold value of the layout may be determined. Knowledge of the cost factors in Equation 42 permits the computation of the cumulative value of the probabilities, $1 - H(x_n)$, which may be associated directly with the threshold value. As soon as the optimization procedure has yielded a layout value equal or less than the threshold point, the application of the decision rule above would indicate that further trials would not be feasible.

Examples of this computation are carried out in Appendix E for the Hillier and Air Force data assuming hypothetical values for C_F and C_c.

By having these decision criteria available, the decision maker is permitted to perform his function more effectively in this challenging area of layout design. He is better prepared to carry out the challenge of Emerson (12, p. 24) "By appropriate comparisons of what is and what ought to be, efficiencies, both ideal and practical, can be established."

BIBLIOGRAPHY

1. Armour, Gordon C. and Elwood S. Buffa. A neuristic algorithm and simulation approach to relative location of facilities. Management Science 9, No. 2: 294-309. 1963.

2. Atkinson, J. D. TARSIER Reference Manual. Numerical Analysis Programming Series No. 8. Statistical Laboratory, Iowa State University, Ames, Iowa. 1966.

3. Bellman, R. E. and Dreyfus, S. E. Applied dynamic programming. R-352-PR. The Rand Corp., Santa Monica, California. 1962.

4. Brooks, Samule H. A discussion of random methods for seeking maxima. Operations Research 6, No. 2: 244-251. 1958.

5. Buffa, Elwood S. Sequence analysis for functional layout. Journal of Industrial Engineering 6, No. 2: 12. 1955.

6. Buffa, Elwood S., Armour, Gordon C., and Vollmann, Thomas E. Allocating facilities with CRAFT. Harvard Business Review 42, No. 2: 136-158. 1964.

7. Churchman, C. West and Ackoff, Russell L. An approximate measure of value. Journal of the Operations Research Society of America 2, No. 2: 172-187. 1954.

8. Conway, R. W. and Maxwell, W. L. A note on the assignment of facility location. Journal of Industrial Engineering 12, No. 1: 34-36. 1961.

9. Dodd, E. L. The greatest and least variate under general laws of error. Trans. Am. Math Soc. 25: 525. 1923.

10. Duncan, Acheson J. Quality control and industrial statistics. Richard D. Irwin, Inc., Homewood, Illinois. 1952.

11. Ebeling, Kenneth. The development of a total systems concept in plant layout. Unpublished paper presented at ASEE Annual Meeting 20-24 June 1966, Washington State University, Pullman, Washington. 1966.

12. Emerson, Harrington. Philosophy of efficiency. The Engineering Magazine 41: 23-26. 1911. Reprinted in classics in Industrial Engineering, Prairie Publishing Co., Delphi, Ind. 1964.

13. Firstmann, Sidney I. and Stoller, David S. Multiphase complimentary activities. Journal of the Operations Research Society of America 14, No. 1: 84-99. 1966.

14. Fisher, R. A. and Tippett, L. H. C. Limiting forms of the frequency distribution of the largest or smallest member of a sample. Proc. Cambridge Phil. Soc. 2, No. 24: 180. 1928.

15. Gavett, J. W. and Norman V. Plyter. The optimal assignments of facilities to locations by branch and bound. Journal of the Operations Research Society of America 14, No. 2: 210-232. 1966.

16. Goode, Henry P. and Kao, John H. K. Sampling procedures and tables for life and reliability testing based on the Weibull distribution (Hazard rate criterion). Bulletin TR-4. Office of the Ass't Secretary of Defense (Installations and Logistics) Washington 25, D.C. 1962.

17. Goode, Henry P. and Kao, John H. K. Sampling procedures and tables for life and reliability testing based on the Weibull distribution (Mean life criterion). Bulletin TR-3. Office of the Ass't Secretary of Defense (Installations and Logistics). Washington 25, D.C. 1961.

18. Granger, Charles H. The hierarchy of objectives. Harvard Business Review 42, No. 3: 63-64. 1964.

19. Gumbel, E. J. Statistical theory of extreme values and some practical applications. National Bureau of Standards, Applied Math Series No. 33. 1954.

20. Gumbel, E. J. Statistics of extremes. Columbia University Press, New York, New York. 1958.

21. Gurland, John. Distribution of the maximum of the arithmetic mean of correlated random variables. Ann. Math. Stats. 26: 294. 1955.

22. Hartley, H. O. The modified Gauss-Newton method for the fitting of non-linear regression functions by least squares. Technometrics 3, No. 2: 269-280. 1961.

23. Henderson, Allen James. The Weibull distribution and industrial property mortality experiences. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa. 1965.

24. Hillier, Frederick S. Quantitative tools for plant layout analysis. Journal of Industrial Engineering 14, No. 1: 33-40. 1963.

25. Hitch, Charles. Sub-optimization in operations problems. Journal of the Operations Research Society of America 1, No. 3: 87-99. 1953.

26. Kao, J. H. K. Computer methods for estimating Weibull parameters in reliability studies. I.R.E. Trans. on Reliability and Quality Control 13: 15-22. 1958.

27. Kiefer, J. Sequential minimax search for a maximum. Proceedings of the American Mathematical Society 4: 502-506. 1953.

28. Koopmans, T. C. and Beckman, M. Assignment problems and location of economic activities. Econometrica 25, No. 1: 53. 1957.

29. Lieblein, Julius. A new method of analyzing extreme-value data. National Advisory Committee for Aeronautics, Technical Note 3053. [National Bureau of Standards. Washington, D.C.] 1954.

30. Lieblein, Julius. On the exact evaluation of the variances and covariances of order statistics in samples from the extreme-value distribution. The Annals of Mathematical Statistics 24, No. 2: 282-287. 1953.

31. Lieblein, Julius. On moments of order statistics from the Weibull distribution. Annals of Math. Stat. 26: 330-333. 1955.

32. Lieblein, Julius and Zelen, M. Statistical Investigation of the fatigue life of deep-groove ball bearings. U.S. National Bureau of Standards, Journal of Research 57, No. 5: 273-316. 1956.

33. Little, John D. C., Murty, Katta G., Sweeney, Dura W., and Karel, Caroline. An algorithm for the traveling salesman problem. Operations Research 11, No. 6: 972-989. 1963.

34. Llewellyn, Robert W. Travel charting with realistic criteria. Journal of Industrial Engineering 9, No. 3: 217-220. 1958.

35. National Bureau of Standards. Probability tables for the analysis of extreme-value data. Applied Math. Series 22. 1953.

36. Noy, Peter C. Make the right plant layout-mathematically. American Machinist 101, No. 6: 62. 1957.

37. Optimum Supply Facility Design. Unpublished briefing report [Prepared by ADC Systems Training and Conversion Team, Richard-Gebaur AFB, Kansas City, Mo., and K. L. McRoberts, Iowa State University.] 1964.

38. Ostle, Bernard. Statistics in research. 2nd ed. Iowa State University Press, Ames, Iowa. 1963.

39. Richman, Eugene. Trends in plant layout and design. Journal of Industrial Engineering 7, No. 1: 29. 1956.

40. Sarin, Lalit K. Introduction to reliability measurement. Report No. 64MAL19. [General Electric Co., Major Applicance Lab., Appliance Park, Louisville, Kentucky]. 1964.

41. Schneider, Marshall. Cross charting technique as a basis for plant layout. Journal of Industrial Engineering No. 6: 478-483. 1960.

42. Scigliano, J. Michael. An evaluation of the Weibull function as an estimator of industrial property mortality. Unpublished M.S. thesis. Library, Iowa State University of Science and Technology, Ames, Iowa. 1965.

43. Simon, H. A. and Newell, H. A. Heuristic problem solving: the next advance in operations research. Journal of the Operations Research Society of America 6, No. 1: 782. 1958.

44. Smith, Wayland P. Travel charting - first aid for plant layout. Journal of Industrial Engineering 6, No. 1: 13. 1955.

45. Starr, M. K. Production management, systems and synthesis. Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.

46. Wagner, H. M., O'Hagan, Michael, and Lundh, Bertil.

An empirical study of exactly and approximately optimal inventory policies. Management Science 11, No. 7: 690-723. 1965.

47. Watson, G. S. Extreme value theory for m-dependent stationary sequences of continuous random variables. Ann. Math. Stats. 25: 798. 1954.

48. Weibull, W. A statistical distribution of wide applicability. Journal of Applied Mechanics 18: 293-297. 1951.

49. Wimmert, R. J. A mathematical method of equipment location. Journal of Industrial Engineering 9, No. 6: 498. 1958.

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This effort is especially dedicated to the memory of H. E. McRoberts, my father, for whom I was six months too slow for sharing this moment.

APPENDIX A

Illustrative Example of the Grid-Flow Procedure

The following example is presented to illustrate the Hillier procedure for finding an idealized arrangement of work centers as programmed on the IBM 360 computer. The example assumes six work centers, each of equal size, and each of which are interrelated with all other work centers by a flow of materials. The criteria which must be satisfied for an idealized arrangement, or layout, is the minimization of the sum of the products of material flow and distance between each of the work centers.

In this procedure, distance is considered as a total of unit distances connecting the mid-points of each work center along an "aisle" path. By an "aisle" path, it is assumed that only right angle patterns are considered in going from the mid-point of one work center to the mid-point of any other. No diagonal paths are considered. The use of a unit distance is permitted only by the critical assumption of identically sized work centers. A variation of this assumption is discussed in a preceeding section.

For this example, the work centers are coded by letters A through F. The following Table 9 represents the number of equivalent material loads within a fixed time period. The unit load equivalence instruction is the second major assumption on which the procedure is developed. The importance of these two assumptions is inherent in the minimization criteria. A load-distance factor of 5 between centers A and F indicates an improved layout relative to these two work centers only if this factor is lesser, on an absolute basis as well as a relative basis, than the load-distance factor between A and J in an alternate layout.

Table 9. Number of equivalent loads per fixed time period between work centers

From	To Work Center					
Work Center	A	B	C	D	E	F
A	-	2	3	5	4	5
B	3	-	2	8	1	0
C	4	0	-	2	5	1
D	0	0	2	-	2	5
E	2	4	3	1	-	5
F	9	4	5	0	1	-

Presumably the six work centers could be arranged in any geometric pattern. The lowest cost factor, in terms of the workload-distance criteria, could occur when all six are lined up in a row or when they are grouped into a rectangular form as close to a square as possible. This example will explore the layout approaching a square, as illustrated in the Figure 14 below, based on the assumption that the absolute minimum will result from a square configuration.

A B C

D E F

Figure 14. Possible initial layout of work centers A-F

Within the initial layout, the distance from A to E is 2 units and from A to F is 3 units. Considering all possible interactions between work centers, the load-distance total for this arrangement is determined as shown in Table 10.

Table 10. Load-distance determination

Load Matrix						X	Distance Matrix							Load-Distance Matrix					
0	2	3	5	4	5		0	1	2	1	2	3		36	X	X	X	X	X
3	0	2	8	1	0		1	0	1	2	1	2		X	22	X	X	X	X
4	0	0	2	5	1	X	2	1	0	3	2	1	=	X	X	25	X	X	X
0	0	2	0	2	5		1	2	3	0	1	2		X	X	X	18	X	X
2	4	3	1	0	5		2	1	2	1	0	1		X	X	X	X	20	X
9	4	5	0	1	0		3	2	1	2	1	0		X	X	X	X	X	41

Total: L-D factor = 36 + 22 + 25 + 18 + 20 + 41
= 162

Since the distance matrix will always by symmetrical about the major diagonal, the major diagonal of the product matrix supplies the meaningful information required. The C_{11} factor in the product matrix is the sum of products of all loads and distances from A to every other center, C_{22} is the sum of products from B, etc. The total load-factor for the layout is the sum across the diagonal factors in the product matrix. This is the criterion to be minimized for an optimum layout.

To reduce the optimum layout criterion the problem assumes the characteristic of a sequencing type of allocation problem to which no general solution technique exists. To make all possible interchanges in the layout on a trial and error basis would require an enormous number of evaluations to make. To make all possible exchanges on a 1 for 1 basis for n work centers would require n! evaluations. For these six example centers this would amount to 720 evaluations. For twelve centers the number of evaluations grows to 479,000,000. Even then the absolute minimum is not assured since multiple trades, 2 for 2, 3 for 3, etc., would also have to be evaluated.

The Hillier procedure is developed on the basis of investigating the 1 for 1 exchanges and evaluating them on a guided basis, thus eliminating the necessity of evaluating all of the alternate solutions. A move effect table can be developed by considering the net effects of moving each work center one, two, or 2 spaces on the total load. For example, by moving A one space to the right, it would be one unit distance closer to all centers to the right of its present position, i.e., B, C, E, F, and one unit distance further away from all centers in its present column and all centers to the left, i.e., D. The net effect would be as in Table 11.

Table 11. Effect on L-D criteria of moving A one space to the right

Criteria	Decrease		Increase
(Center)	(L-D)	(Center)	(L-D)
B	1x(2+3)	D	1x(5+0)=5
C	1x(3+4)=7		
E	1x(4+2)=6		
F	1x(5+9)=14		
Total	32		5
Net effect = 27 decrease			

One alternative exchange that might be made by moving A one space to the right is moving B one space to the left, or making a 1 for 1 exchange of A and B. Moving B to the left would decrease the L-D criteria by the amount of the load to A and to D and increase the criteria by the load to C, E, and F. The effect of this move is a decrease of 2 units in the layout criteria. Thus if A and B are exchanged the net effect on the criteria is as follows:

$$\Delta L\text{-}D_{AB} = \Delta A_{R_1} + \Delta B_{L_1} - 2(AB)$$

$$= 27 + 2 - (2)(2+3) = 19 \qquad (43)$$

The change in the L-D factor is equal to the change by moving A to the right one space and B to the left one space. Since A and B are exchanged their relative positions to each other will cause no change in the criteria due to the load flow between them. Since this load flow was included in both the right and left changes of A and B, it will have to be subtracted out for the proper net change to be computed.

By leaving the changes right and left of all centers unaltered, even though it causes the center to be "overlaid" on another center, it is possible to consider all diagonal moves as well, i.e., A and E exchange. This latter would be evaluated by moving A right one space and down one space additively, E up and left one space additively, and subtracting out the load flow between A and E. Examples of the computation and the resulting move effects are displayed in Tables 12, 13, and 14.

Table 12. Example of computing the move effects for each work center moved one space to the right

	Unit Change[a]							Total Load							Effect[b]					
	A	B	C	D	E	F		A	B	C	D	E	F		A	B	C	D	E	F
A	0	+1	+1	-1	+1	+1		0	5	7	5	6	14		27					
B	-1	0	+1	-1	-1	+1		5	0	2	8	5	4			-12				
C	0	0	0	0	0	0		7	2	0	4	8	6				0			
D	-1	+1	+1	0	+1	+1	X	5	8	4	0	3	5	=				15		
E	-1	-1	+1	-1	0	+1		6	5	8	3	0	6						0	
F	0	0	0	0	0	0		14	4	6	5	6	0							0

[a]A positive notation indicates that the work center moved is 1 unit closer to the alternate work center and hence will result in a decrease to the L-D criterion, i.e., moving A one space right moves it +1 unit closer to B.

[b]The major diagonal include the only meaningful values and indicate the effect of moving the associated work center one space to the right.

Table 13. Initial move effect table[a]

Work Center	Right 1	Right 2	Left 1	Left 2	Up	Down
A	27	32	0	0	0	13
B	-12	0	2	0	0	10
C	0	0	15	10	0	9
D	15	8	0	0	9	0
E	0	0	-10	0	10	0
F	0	0	23	26	13	0

[a]The negative sign notation indicates a decrease in the L-D criteria.

By arranging the centers again in matrix form, with the elements indicating the net effect of the exchanges between the row and column centers which the elements intersect, the unadjusted criterion change can be determined. From this, the matrix of double the load values can be subtracted leaving a matrix of values indicating in total the increase or decrease to the original layout criterion resulting from the 1 for 1 exchange. The largest value indicates the change that must be made.

Example Calculation: A - E exchange

(A right 1) + (A down 1) + (E left 1) + (E up 1)

$$27 + 13 + (-10) + (10) = 40$$

(2)(Total load between A and E)(No. units apart)

$$= (2)(6)(2) = 24$$

$$\Delta(L\text{-}D) = 40 - 24 = 16$$

Table 14. Computation of Δ(L-D)

[Unadjusted effect] - [(2)(Total Load)(Number Units Apart)] = Δ(L-D)

	A	B	C	D	E	F		A	B	C	D	E	F		A	B	C	D	E	F
A	0	29	42	22	40	84		0	10	28	10	24	84		0	19	14	12	16	0
B		0	3	36	20	34			0	4	32	10	16			0	-1	4	10	18
C			0	36	34	22				0	24	32	12				0	12	2	10
D				0	5	34	-				0	6	20	=				0	-1	14
E					0	23						0	12						0	11
F						0							0							0

The L-D criterion for the initial layout with only A and E exchanged would equal 162-16 = 146.

From the Δ(L-D) matrix in Table 14, the greatest improvement to the initial layout will result in an exchange of work centers A and B. This will result in a reduction of the L-D criterion of 19. This process must now be repeated with each successive layout improvement until all values in the Δ(L-D) matrix are 0 or negative indicating that any further 1 for 1 exchanges will result in no improvement or will lead to a worse layout in terms of a higher L-D factor.

In this example the lowest L-D factor will occur in the fourth iteration at a value of 129. The resulting arrangement will be as shown in Figure 15.

D	A	C
B	F	E

Figure 15. Layout with lowest L-D criterion

This, however, is the result of one trial solution. A lower criterion value may result if the centers could be interchanged 2 for 2, or higher multiple exchanges. To investigate this, a simulation process is introduced whereby a series of random starting layout are generated and each is processed through to its lowest value, this generating a distribution of lowest values.

Appendix B illustrates the machine print-out for the above example problem.

APPENDIX B

IBM 360 output for example problem in Appendix A.

ISU COMPUTATION CENTER; AMES, IOWA

PLANT LAYOUT ANALYSIS HILLIER ALGORITHM

```
 1    2    3
 4    5    6

COST=      162      NN=      1

 4    1    3
 2    6    5

COST=      129      NN=      1
```

IBM 360 Time Requirement: approximately 2 seconds

APPENDIX C

Table 15. Cumulative probability table for the Weibull distribution

$$P(x \le x) = 1 - e^{-A^k} \qquad A = [\frac{x-\varepsilon}{V-\varepsilon}]$$

Λ \ k	0.5	1.0	1.5	2.0	2.5
0.1	0.2711	0.0952	0.0311	0.0100	0.0032
0.2	0.3606	0.1813	0.0856	0.0392	0.0177
0.3	0.4217	0.2592	0.1515	0.0861	0.0481
0.4	0.4687	0.3297	0.2235	0.1479	0.0962
0.5	0.5069	0.3935	0.2978	0.2212	0.1620
0.6	0.5391	0.4512	0.3717	0.3023	0.2434
0.7	0.5668	0.5034	0.4433	0.3874	0.3363
0.8	0.5912	0.5507	0.5111	0.4727	0.4358
0.9	0.6127	0.5934	0.5742	0.5551	0.5363
1.0	0.6321	0.6321	0.6321	0.6321	0.6321
1.1	0.6496	0.6671	0.6845	0.7018	0.7189
1.2	0.6656	0.6988	0.7314	0.7631	0.7935
1.3	0.6802	0.7275	0.7729	0.8155	0.8544
1.4	0.6937	0.7534	0.8092	0.8591	0.9016
1.5	0.7062	0.7769	0.8407	0.8946	0.9364
1.6	0.7177	0.7981	0.8679	0.9227	0.9608
1.7	0.7285	0.8173	0.8910	0.9444	0.9769
1.8	0.7386	0.8347	0.9106	0.9608	0.9871
1.9	0.7480	0.8504	0.9271	0.9729	0.9931
2.0	0.7569	0.8647	0.9409	0.9817	0.9965
2.1	0.7652	0.8775	0.9523	0.9878	0.9983
2.2	0.7731	0.8892	0.9617	0.9921	0.9992
2.3	0.7805	0.8997	0.9694	0.9950	0.9997
2.4	0.7876	0.9093	0.9757	0.9968	0.9999
2.5	0.7943	0.9179	0.9808	0.9981	0.9999
2.6	0.8006	0.9257	0.9849	0.9988	1.0000
2.7	0.8066	0.9328	0.9882	0.9993	
2.8	0.8124	0.9392	0.9908	0.9996	
2.9	0.8179	0.9450	0.9928	0.9998	
3.0	0.8231	0.9502	0.9945	0.9999	
3.1	0.8281	0.9550	0.9957	0.9999	
3.2	0.8328	0.9592	0.9967	1.0000	
3.3	0.8374	0.9631	0.9975		
3.4	0.8418	0.9666	0.9981		
3.5	0.8460	0.9698	0.9986		
3.6	0.8500	0.9727	0.9989		
3.7	0.8539	0.9753	0.9992		
3.8	0.8576	0.9776	0.9994		
3.9	0.8612	0.9798	0.9995		
4.0	0.8647	0.9817	0.9997		

Table 15 (Continued)

$$P(x \leq x) = 1 - e^{-A^k} \qquad A = \left[\frac{x-\varepsilon}{v-\varepsilon}\right]$$

A \ k	3.0	3.5	4.0	4.5	5.0
0.1	0.0010	0.0003	0.0001	0.0000	0.0000
0.2	0.0080	0.0036	0.0016	0.0007	0.0003
0.3	0.0266	0.0147	0.0081	0.0044	0.0024
0.4	0.0620	0.0397	0.0253	0.0161	0.0102
0.5	0.1175	0.0846	0.0606	0.0432	0.0308
0.6	0.1943	0.1541	0.1216	0.0955	0.0748
0.7	0.2904	0.2495	0.2135	0.1820	0.1547
0.8	0.4007	0.3674	0.3361	0.3067	0.2794
0.9	0.5176	0.4992	0.4811	0.4634	0.4459
1.0	0.6321	0.6321	0.6321	0.6321	0.6321
1.1	0.7358	0.7524	0.7687	0.7847	0.8002
1.2	0.8224	0.8494	0.8743	0.8968	0.9170
1.3	0.8889	0.9183	0.9425	0.9615	0.9756
1.4	0.9357	0.9611	0.9785	0.9894	0.9954
1.5	0.9658	0.9840	0.9937	0.9980	0.9995
1.6	0.9834	0.9944	0.9986	0.9997	1.0000
1.7	0.9926	0.9983	0.9998	1.0000	
1.8	0.9971	0.9996	1.0000		
1.9	0.9990	0.9999			
2.0	0.9997	1.0000			
2.1	0.9999				
2.2	1.0000				

APPENDIX D

Transformation of First Asymptote to the Third Asymptote

The general form of the first asymptote is:

$$\psi(x) = e^{-e^{-\alpha(x-u)}} \tag{44}$$

let

$$\alpha = k \tag{45}$$

$$x - u = -\ln\left(\frac{x-\epsilon}{v-\epsilon}\right) \tag{46}$$

then

$$\ln Y(x) = -e^{-\alpha(x-u)} \tag{47}$$

$$\ln\ln Y(x) = \alpha(x-u) \tag{48}$$

and by substitution of (2) and (3)

$$\ln\ln Y(x) = k\left(-\ln\frac{x-\epsilon}{v-\epsilon}\right) \tag{49}$$

$$\ln Y(x) = -\left(\frac{x-\epsilon}{v-\epsilon}\right)^k \tag{50}$$

$$Y(x) = e^{-\left(\frac{x-\epsilon}{v-\epsilon}\right)^k} = H(x) \tag{51}$$

APPENDIX E

Example Computation for Justification of Additional Design

A. Hillier Data

Assume that the following estimates are applicable to the layout values provided from the Hillier data.

a) C_F = \$800

b) A computer trial requires 2 sec.

c) Costs associated with the computer trials total \$200/hour.

1)

x_i	Col. 1 $1-H(x_i)$	Col. 2 $\rho(x_i-x_{i-1})$	Col. 3 x_i-x_{i-1}	Col. 2xCol. 3
282	0	0	1	0
283	~0	0	1	0
284	~0	0	1	0
285	~0	0	1	0
286	.0001	.0001	1	.0001
287	.002	.0019	1	.0019

$$1-H(287) = \sum_{282}^{287}[\rho(x_i-x_{i-1})][x_i-x_{i-1}] = .002$$

2) $C_c = (2)[\frac{200}{3600}] = 0.1110$

3) $\frac{C_F}{C_c}[1-H(287)] = [\frac{800}{0.1110}][0.002]$
$= 14 +$

Decision: Make additional trial.

4) Threshold value of $x = x_T$

$$1 - H(x_T) = \frac{C_c}{C_F}$$

$$= \frac{0.1110}{800} = 0.000139$$

$$H(x_T) = 0.999861$$

$$e^{-\left[\frac{x_T - 282}{326-282}\right]^{2.96}} = 0.999861$$

$$- \frac{(x_T - 282)^{2.96}}{(44)^{2.96}} = -0.0008$$

$$[x_T - 282]^{2.96} = (44)^{2.96}\ (0.0008)$$

$$x_T - 282 = (44)(0.090)$$

$$x_T = 286$$

Decision: Continue trials until layout value of 286 is achieved.

B. U.S.A.F. Data

Assume the following cost estimates.

a) $C_F = 800$

b) A computer trial requires 8 sec.

c) Costs associated with the computer trials total $200/hour.

1) $1 - H(x_n) = 1 - H(340) = 0.0602$

2) $C_c = (8)(\frac{200}{3600}) = 0.445$

3) $\frac{C_F}{C_c}\,[1 - H(340)] = [\frac{800}{0.445}]\ 0.0602$
$= 108$

Decision: Make additional trial.

4) Threshold value of $x = x_T$

$$1 - H(x_T) = \frac{C_c}{C_F}$$

$$= \frac{0.445}{800} = .00056$$

$$H(x_T) = 0.99944$$

$$e^{-\left(\frac{x_T-325}{398-325}\right)^{1.76}} = 0.99944$$

$$-\frac{(x_T-325)^{1.76}}{(73)^{1.76}} = -0.0004$$

$$x_T - 325 = (73)(.012)$$

$$x_T = 326$$

Decision: Continue trials until layout value of 326 is achieved.

APPENDIX F: "WEIBULL" PROBABILITY PAPER